

$L^p \rightarrow L^q$ REGULARITY OF FOURIER INTEGRAL OPERATORS WITH CAUSTICS

ANDREW COMECH

ABSTRACT. The caustics of Fourier integral operators are defined as caustics of the corresponding Schwartz kernels (Lagrangian distributions on $X \times Y$). The caustic set $\Sigma(\mathbf{C})$ of the canonical relation is characterized as the set of points where the rank of the projection $\pi : \mathbf{C} \rightarrow X \times Y$ is smaller than its maximal value, $\dim(X \times Y) - 1$. We derive the $L^p(Y) \rightarrow L^q(X)$ estimates on Fourier integral operators with caustics of corank 1 (such as caustics of type A_{m+1} , $m \in \mathbb{N}$). For the values of p and q outside of a certain neighborhood of the line of duality, $q = p'$, the $L^p \rightarrow L^q$ estimates are proved to be caustics-insensitive.

We apply our results to the analysis of the blow-up of the estimates on the half-wave operator just before the geodesic flow forms caustics.

Stretching his stiff legs, Persikov got up, returned to his laboratory, yawned, rubbed his permanently inflamed eyelids, sat down on the stool and looked into the microscope . . . With his right eye Persikov saw the cloudy white plate and blurred pale amoebas on it, but in the middle of the plate sat a coloured tendril, like a female curl. The coloured streak of light merely got in the way and indicated that the specimen was out of focus. The zoologist's long fingers had already tightened on the knob, when suddenly they trembled and let go . . . He noticed that one particular ray in the coloured tendril stood out more vividly and boldly than the others . . . This strip of red was teeming with life. The old amoebas were forming pseudopodia in a desperate effort to reach the red strip, and when they did they came to life, as if by magic. They split into two in the ray, and each of the parts became a new, fresh organism in a couple of seconds.

Mikhail Bulgakov, *The Fateful Eggs*, 1924.

1. INTRODUCTION

Caustics are the envelopes of the light rays. At the caustic points, intensity of light is singularly large, causing different physical phenomena (such as the one observed by Professor Persikov). Mathematically, caustics could be characterized as points where usual bounds on oscillatory integrals are no longer valid. In this paper we are going to consider how this concept applies to Fourier integral operators. This question becomes interesting in view of a recent paper [JMR00] on dissipative semilinear oscillations, where the L^q estimates on oscillatory integrals with caustics played the central role. Our goal is to investigate how the regularity properties of Fourier integral operators are affected by the presence of caustics. We will show, in particular, that for q away from a certain neighborhood of $q = p'$ the $L^p \rightarrow L^q$ bounds on Fourier integral operators are caustic-insensitive.

Received by the editors January 22, 2003.

2000 *Mathematics Subject Classification.* Primary 35S30.

This work was supported in part by the NSF under Grants No. 0296036 and 0200880.

Oscillatory integrals with caustics have enjoyed much attention. The classical references are [AGZV88] and [Dui74]. The asymptotics of oscillatory integrals near caustics were derived in [Lud66] and [GS77].

Let us mention previously known estimates on Fourier integral operators. The L^2 estimates on Fourier integral operators were considered by Hörmander [Hör71]. The $L^p \rightarrow L^q$ and $L^p \rightarrow L^{p'}$ estimates on Fourier integral operators in the context of strictly hyperbolic equations with constant coefficients were addressed in [Str70], [Lit73], [Bre75], and [Sug94, Sug96, Sug98]. $L^p \rightarrow L^{p'}$ estimates for certain hyperbolic equations with smooth coefficients and applications to the existence and uniqueness results for semi-linear hyperbolic equations are in [Bre77]. The $L^p \rightarrow L^p$ estimates were derived by Seeger, Sogge, and Stein [SSS91]. For more information on regularity properties of generalized Radon transforms and Fourier integral operators associated to local graphs and to degenerate canonical relations see the reviews [GSW00, GS02].

We first recall some background about caustics of oscillatory integrals. Let us consider an oscillatory integral

$$(1.1) \quad u_\tau(x) = \tau^{k/2} \int_{\mathbb{R}^k} e^{i\tau\psi(x,\alpha)} a(x,\tau,\alpha) d^k\alpha, \quad x \in \mathbb{R}^n, \quad \alpha \in \mathbb{R}^k, \quad \tau > 0.$$

We assume that ψ is a smooth function and that $a \in S^d$ is a symbol of order d in τ , compactly supported in α and x . If there are no critical points of the map $\alpha \mapsto \psi(x,\alpha)$, so that $\psi'_\alpha \neq 0$ everywhere in an open neighborhood of the support of $a(x,\tau,\alpha)$, then the repeated integration by parts shows that $|u_\tau(x)| = o(\tau^{-N})$, for any $N > 0$. If there are non-degenerate critical points, where $\psi'_\alpha = 0$ but $\det \psi''_{\alpha\alpha} \neq 0$, then the method of stationary phase shows that $|u_\tau(x)| = O(\tau^d)$. If we also assume that $\text{rank } \psi''_{x\alpha} \geq n$ (when $k \geq n$), then one can readily show that $\|u_\tau(x)\|_{L^2} = O(\tau^d)$. It follows that as long as all critical points are non-degenerate, $u_\tau(x) \in L^q(\mathbb{R}^n)$, $2 \leq q \leq \infty$, with the norms bounded uniformly in $\tau \in (0, \infty)$.

If there are degenerate critical points, known as caustics, then $\|u_\tau(x)\|_{L^\infty}$ is no longer uniformly bounded. The order of a caustic κ is defined as the infimum of κ' so that $\|u_\tau(x)\|_{L^\infty} = O(\tau^{\kappa'})$. For example, $\psi(x,\alpha) = \alpha^3 + x\alpha$ corresponds to the fold (A_2), with $\kappa = 1/6$; $\psi(x,\alpha) = \alpha^4 + x_1\alpha^2 + x_2$ corresponds to the cusp (A_3), with $\kappa = 1/4$. For more details, see [AGZV88], [Dui74], [Dui96]. At the same time, it was shown in [JMR00] that there exists $q_c > 2$ such that the L^q estimates for $2 \leq q < q_c$ are still bounded uniformly in τ . (This information was used to deduce that the singularities of solutions to dissipative semilinear equations, like $\square u = \dot{u}|\dot{u}|^{p-1}$, are absorbed at the caustic if p is larger than certain critical value; for generic caustics, one needs $p \geq 3$ for such an absorption to take place.)

Now we turn to Fourier integral operators. Let X and Y be two smooth manifolds (without boundary). A Fourier integral operator $\mathfrak{F} : C_0^\infty(Y) \rightarrow \mathcal{D}'(X)$ can be defined (locally) by

$$(1.2) \quad \mathfrak{F}u(x) = \int_{\mathbb{R}^N \times Y} e^{i\phi(x,\theta,y)} a(x,\theta,y) u(y) d\theta dy,$$

where a is a symbol of order d and ϕ is a non-degenerate phase function. We write $\mathfrak{F} \in I^\mu(X, Y, \mathbb{C})$, where the order of the operator is defined by

$$\mu = d + \frac{N}{2} - \frac{\dim X + \dim Y}{4}$$

and \mathbf{C} is the associated canonical relation. We will always assume that

$$\dim X = \dim Y = n$$

and that the symbol a is compactly supported in $X \times Y$.

Let us consider $L^1 \rightarrow L^\infty$ estimates on \mathfrak{F} . From (1.2) one can see that

$$(1.3) \quad \mathfrak{F} : L^1(Y) \rightarrow L^\infty(X)$$

if $d + N < 0$ (which is equivalent to $\mu < -(n + N)/2$). The smaller the minimal number of oscillatory variables is, the better $L^1 \rightarrow L^\infty$ regularity properties \mathfrak{F} possesses. As we know from [Hör71], the minimal number of oscillatory variables is equal to $N_{\min} = 2n - r$, where r is the minimal value of the rank of the projection $\pi_{X \times Y}$ from \mathbf{C} onto $X \times Y$:

$$r = \min \text{rank } d\pi_{X \times Y}.$$

We define the caustic set of the canonical relation as a subset $\Sigma(\mathbf{C})$ of \mathbf{C} where the rank of $d\pi_{X \times Y}$ is not maximal:

$$\Sigma(\mathbf{C}) = \{p \in \mathbf{C} : \text{rank } d\pi_{X \times Y}|_{\Sigma(\mathbf{C})} < 2n - 1\},$$

so that outside of $\Sigma(\mathbf{C})$ the number of oscillatory variables of a Fourier integral operator $\mathfrak{F} \in I^\mu(X, Y, \mathbf{C})$ could be reduced to $N = 1$. Let $\mathfrak{F} \in I^\mu(X, Y, \mathbf{C})$, and let \mathfrak{F}_λ , $\lambda \geq 1$, be its Littlewood-Paley decomposition. Similarly to [Dui96], we will say that κ is the highest order of caustics of \mathbf{C} if it is the infimum of numbers κ' such that the Schwartz kernel of \mathfrak{F}_λ , which is an oscillatory function of order μ , is bounded by $O(\lambda^{\mu+\kappa'})$, uniformly in x and y . It follows that for the action

$$\mathfrak{F} : L^1(Y) \rightarrow L^\infty(X)$$

to be continuous we need to have $\mathfrak{F} \in I^\mu$ with $\mu < -\frac{n+1}{2} - \kappa$. Thus, in the presence of caustics, the $L^1 \rightarrow L^\infty$ estimates deteriorate. On the other hand, if we assume that \mathbf{C} is a local graph, the mappings

$$(1.4) \quad \mathfrak{F} : L^2(Y) \rightarrow L^2(X) \quad \text{if } \mu \leq 0,$$

$$(1.5) \quad \mathfrak{F} : \mathfrak{h}^1(Y) \rightarrow L^1(X) \quad \text{if } \mu \leq -\frac{n-1}{2},$$

$$(1.6) \quad \mathfrak{F} : \mathfrak{h}^1(Y) \rightarrow L^2(X) \quad \text{if } \mu \leq -\frac{n}{2},$$

are continuous, independently of the presence of caustics. (The estimate (1.4) is the classical L^2 bound on Fourier integral operators, (1.5) is proved in [SSS91], and (1.6) follows from the $\mathfrak{h}^1 \rightarrow L^\infty$ estimate on $\mathfrak{F}\mathfrak{F}^*$, which is a pseudodifferential operator of order 2μ .) The $L^p \rightarrow L^q$ estimates for $1 < p \leq q \leq 2$ (obtained by interpolation of (1.4), (1.5), and (1.6)) are also caustic-insensitive. By duality considerations, the same is true for $2 \leq p \leq q < \infty$. We are going to show that for $1 < p \leq 2 \leq q < \infty$ away from a certain neighborhood of the line $q = p'$ the $L^p \rightarrow L^q$ estimates are also caustic-insensitive. In this paper, we only consider the situation when $\text{rank } d\pi_{X \times Y} \geq 2n - 2$.

Let us give a short account of our methods. Let $\mathbf{C} \subset T^*(X) \setminus 0 \times T^*(Y) \setminus 0$ be a canonical relation which is a local graph. Let $\mathfrak{F} = \sum_\lambda \mathfrak{F}_\lambda + \mathfrak{F}_0$, $\lambda = 2^k$, $k \in \mathbb{N}$, be a Littlewood-Paley decomposition of $\mathfrak{F} \in I^\mu(X, Y, \mathbf{C})$. If the caustic set $\Sigma(\mathbf{C})$ is empty ($\text{rank } d\pi_{X \times Y} = 2n - 1$ everywhere), then, representing \mathfrak{F} with $\theta \in \mathbb{R}^1$, one easily checks that

$$(1.7) \quad \|\lambda^{\mu+\frac{n+1}{2}} \mathfrak{F}_\lambda\|_{L^1 \rightarrow L^\infty} < C,$$

uniformly in λ . Now let $\Sigma(\mathbf{C}) \neq \emptyset$, and assume that $\text{rank } d\pi_{X \times Y} = 2n - 2$ at $\Sigma(\mathbf{C})$. Choosing local coordinates α on the unit sphere in \mathbb{R}^N , we introduce a function

$$(1.8) \quad \mathcal{D} = \det_{ij}(|\theta|^{-1} \phi''_{\alpha_i \alpha_j}), \quad 1 \leq i, j \leq N - 1,$$

which measures the distance to the caustic set. This function is defined up to a nonzero factor, which depends on the local coordinates. We decompose \mathfrak{F}_λ into $\sum_\sigma \mathfrak{F}_{\lambda, \sigma} + \mathfrak{F}_{\lambda, \text{nice}}$, where $\sigma = 2^{-j}$, $j \in \mathbb{N}$, with respect to the values of \mathcal{D} , so that the Schwartz kernel of $\mathfrak{F}_{\lambda, \sigma}$ is localized to the set $\sigma/2 \leq |\mathcal{D}| \leq 2\sigma$ near $\Sigma(\mathbf{C})$, while the Schwartz kernel of $\mathfrak{F}_{\lambda, \text{nice}}$ is localized away from $\Sigma(\mathbf{C})$. When approaching the caustic set, the $L^1 \rightarrow L^\infty$ estimates become worse:

$$(1.9) \quad \|\lambda^{-\mu - \frac{n+1}{2}} \mathfrak{F}_{\lambda, \sigma}\|_{L^1 \rightarrow L^\infty} \sim \sigma^{-1/2}.$$

This is the optimal estimate which one expects from the application of the stationary phase method. On the other hand, the $L^1 \rightarrow L^2$ action of $\mathfrak{F}_{\lambda, \sigma}$ improves near $\Sigma(\mathbf{C})$:

$$(1.10) \quad \|\lambda^{-\mu - \frac{n}{2}} \mathfrak{F}_{\lambda, \sigma}\|_{L^1 \rightarrow L^2} \sim \sigma^{\frac{1}{2m}}.$$

Using the idea from [Tom79], this estimate is essentially the “square root” of the estimate on the $L^1 \rightarrow L^\infty$ action of $\lambda^{-2\mu-n} \mathfrak{F}_{\lambda, \sigma}^* \mathfrak{F}_{\lambda, \sigma}$. The Schwartz kernel of $\lambda^{-2\mu-n} \mathfrak{F}_{\lambda, \sigma}^* \mathfrak{F}_{\lambda, \sigma}$ is bounded uniformly in x , y , and λ , and hence this operator is bounded from L^1 to L^∞ (uniformly in λ). Moreover, the Schwartz kernel involves an inert integration in θ , and if \mathcal{D} vanishes of order m with respect to θ (as for the caustics of the type A_{m+1}), then there is an improvement $\|\lambda^{-2\mu-n} \mathfrak{F}_{\lambda, \sigma}^* \mathfrak{F}_{\lambda, \sigma}\|_{L^1 \rightarrow L^\infty} \sim \sigma^{1/m}$ for small values of σ , which leads to (1.10).

The estimates (1.9) and (1.10) allow us to prove that there is some $q_c > 2$ such that for $2 \leq q < q_c$ the $L^1 \rightarrow L^q$ regularity of \mathfrak{F}_λ is not affected by caustics. In essence, this situation is expressed by the following obvious lemma:

Lemma 1.1. *Let B_t , $0 \leq t \leq 1$, be a family of complete Banach spaces and that $B_t \subset B_{t'}$, $\|\cdot\|_{B_t} \geq \|\cdot\|_{B_{t'}}$ if $t \geq t'$. Moreover, assume that for any $\xi \in B_1 \subset B_0$,*

$$\|\xi\|_{B_t} \leq \|\xi\|_{B_0}^{1-t} \|\xi\|_{B_1}^t.$$

Let $\xi_j \in B_1$, $j \in \mathbb{N}$, be a sequence such that $\|\xi_j\|_{B_0} \leq a^j$, $0 < a < 1$, and that $\|\xi_j\|_{B_1} \leq b^j$, $b > 1$. Then $\sum_{j \in \mathbb{N}} \xi_j$ converges in B_t for $0 \leq t < t_c \equiv \frac{-\ln a}{-\ln a + \ln b}$.

This already allows us to calculate the critical value q_c . The estimate (1.9) (mapping to L^∞) blows up as $\sigma^{-1/2}$, while the estimate (1.10) (mapping to L^2) improves as $\sigma^{1/(2m)}$. Interpolation shows that the bound on the $L^1 \rightarrow L^q$ mapping of $\lambda^{-\mu-(n+1)/2+1/q} \mathfrak{F}_{\lambda, \sigma}$ behaves as $\sigma^{-1/2+(1+1/m)/q}$, which improves for small σ if $q < q_c = 2 + 2/m$, so that the mapping

$$\lambda^{-\mu-(n+1)/2+1/q} \mathfrak{F}_\lambda : L^1(Y) \rightarrow L^q(X), \quad 2 \leq q < q_c,$$

is bounded uniformly in λ and is not affected by the caustics. These estimates could be interpolated with the (caustic-insensitive) $L_\mu^2 \rightarrow L^2$ estimates on \mathfrak{F}_λ . For $1 < p \leq 2 \leq q < \infty$, Littlewood-Paley theory implies that the mapping $\mathfrak{F} : L_\alpha^p \rightarrow L^q$ is continuous if the mappings $\lambda^\alpha \mathfrak{F}_\lambda : L^p \rightarrow L^q$ are bounded uniformly in λ . This, together with duality considerations, yields the range of p and q such that the $L^p \rightarrow L^q$ estimates are caustic-insensitive.

Remark 1.2. This situation is similar to the $L^p \rightarrow L^p$ regularity of Fourier integral operators associated to degenerate canonical relations, when the projections from the canonical relation are allowed to have singularities. For example, as we know from [MT85], if both projections $\mathbf{C} \rightarrow T^*(X)$, $\mathbf{C} \rightarrow T^*(Y)$ have at most Whitney fold singularities, then the operator $\mathfrak{F} \in I^\mu(X, Y, \mathbf{C})$ loses $1/6$ of a derivative in the Sobolev spaces: $\mathfrak{F} : H^\alpha(Y) \rightarrow H^{\alpha-\mu-\frac{1}{6}}(X)$, but according to [SS94] the $L^p \rightarrow L^p$ regularity of such an operator for $p \notin (1, 3/2) \cup (3, \infty)$ is the same as for operators associated to local graphs [SSS91]: $\mathfrak{F} : L^\mu_\alpha(Y) \rightarrow L^\mu_{\alpha-\mu-(n-1)|\delta_p|}(X)$, $\delta_p = \frac{1}{p} - \frac{1}{2}$. Similar estimates on operators with one-sided Whitney folds were derived in [CC03]. This time, one uses the Phong-Stein decomposition [PS91] of \mathfrak{F} with respect to the distance to the critical variety Σ (where the projections from the canonical relation become singular). This distance is measured by the function

$$h = |\theta|^{N-n} \det \begin{bmatrix} \phi''_{xy} & \phi''_{x\theta} \\ \phi''_{\theta y} & \phi''_{\theta\theta} \end{bmatrix},$$

which is proportional to the determinants of the Jacobi matrices of projections from \mathbf{C} . (The factor in the definition of h is chosen so that h is homogeneous of degree 0 in θ .) We decompose $\mathfrak{F} = \sum_{\hbar} \mathfrak{F}_{\hbar} + \mathfrak{F}_{\text{smooth}}$, where $\hbar = 2^{-j}$, $j \in \mathbb{N}$. The operator \mathfrak{F}_{\hbar} is obtained from \mathfrak{F} by localizing its integral kernel to the variety where $\hbar/2 \leq |h| \leq 2\hbar$, and the projections from \mathbf{C} have no singularities on the support of the integral kernel of $\mathfrak{F}_{\text{smooth}}$. The main observation is that while near Σ the Sobolev estimates become worse, $\|\mathfrak{F}_{\hbar}\|_{L^2_\mu \rightarrow L^2} \sim \hbar^{-1/2}$, the Hardy space to L^1 estimates improve due to smaller size of the support of the integral kernel: $\|\mathfrak{F}_{\hbar}\|_{\mathfrak{h}^1_{\mu+\frac{n-1}{2}} \rightarrow L^1} \sim \hbar$.

Caustics of Lagrangian distributions are discussed in Section 2. The main results (Theorems 3.5 and 3.11) are stated in Section 3. The proof of $L^p \rightarrow L^q$ estimates is in Section 4. The sharp $\mathfrak{h}^1 \rightarrow L^q$ estimates are proved in Section 5. We apply our results to the estimates on the half-wave operator in Section 6.

The consistency of definition (1.8) of the distance \mathcal{D} to the caustic set is proved in Appendix A. The technical lemma ($\mathfrak{h}^1 \rightarrow L^\infty$ bounds on pieces) which allows us to obtain $\mathfrak{h}^1 \rightarrow L^q$ estimates is proved in Appendix B.

2. CAUSTICS OF LAGRANGIAN DISTRIBUTIONS

2.1. Symbols. We will use the class of classical (polyhomogeneous) symbols, in the sense of [Hör94].

Definition 2.1. A smooth function $a(x, \theta)$ on $X \times \mathbb{R}^N$ is called a symbol of order d in θ if for any multi-indices $\alpha \in \mathbb{Z}_+^n$ and $\beta \in \mathbb{Z}_+^N$

$$|\partial_x^\alpha \partial_\theta^\beta a(x, \theta)| \leq C_{\alpha, \beta} (1 + |\theta|)^{d-|\beta|}, \quad \text{for all } (x, \theta) \in X \times \mathbb{R}^N,$$

where $|\beta| = \beta_1 + \cdots + \beta_N$.

We denote the class of symbols of order d by $S^d(X \times \mathbb{R}^N)$ or simply by S^d .

The class of classical (or polyhomogeneous) symbols $S^d_{\text{cl}}(X \times \mathbb{R}^N)$ consists of symbols $a(x, \theta) \in S^d(X \times \mathbb{R}^N)$ that satisfy an asymptotic development of the form

$$(2.1) \quad a(x, \theta) \sim \sum_{j=0}^{\infty} a_j(x, \theta),$$

where a_j are smooth functions on $X \times \mathbb{R}^N$ positively homogeneous of degree $d - j$ for $|\theta| \geq 1$:

$$(2.2) \quad a_j(x, \tau\theta) = \tau^{d-j} a_j(x, \theta) \quad \text{if } |\theta| \geq 1, \quad \tau \geq 1.$$

The asymptotic development (2.1) means that we have

$$(2.3) \quad a(x, \theta) - \sum_{j=0}^{k-1} a_j(x, \theta) = O(|\theta|^{d-k}) \quad \text{for } |\theta| \geq 1$$

and similar estimates for the derivatives.

2.2. Oscillatory functions. Let X be a C^∞ manifold and $\Lambda \subset T^*(X)$ a C^∞ Lagrangian submanifold. We say that $\psi \in C^\infty(X \times \mathbb{R}^k)$ parametrizes Λ (locally) if

$$(2.4) \quad d_{(x,\alpha)} d_\alpha \psi \quad \text{has rank } k \text{ when } d_\alpha \psi = 0$$

and Λ is locally given by

$$(2.5) \quad \Lambda_\psi = \{(x, d_x \psi(x, \alpha)) : d_\alpha \psi(x, \alpha) = 0\}.$$

Definition 2.2. Let Λ be a C^∞ Lagrange submanifold in $T^*(X)$. An oscillatory function $u(x, \tau)$ of order μ defined by Λ is a locally finite (in X) sum of integrals of the form

$$I(x, \tau) = \int e^{i\tau\psi(x, \tau, \alpha)} b(x, \tau, \alpha) d\alpha,$$

where $\alpha = (\alpha_1, \dots, \alpha_k)$, $k \in \mathbb{N}$, ψ satisfies (2.4), Λ_ψ is a piece of Λ and $b(x, \tau, \alpha) \in S_{\text{cl}}^{\mu + \frac{k}{2}}$ (a classical symbol of order $\mu + \frac{k}{2}$ in τ) vanishes for α outside a fixed compact set in \mathbb{R}^k .

Definition 2.3 (Duistermaat [Dui96]). Let $i : \Lambda \rightarrow T^*(X)$ be an immersed Lagrange manifold in $T^*(X)$. The caustic $c(\Lambda)$ of Λ is the projection into X of the set $\Sigma(\Lambda)$ of points in $i(\Lambda)$ where i is not transversal to the fibers. At each point $x_0 \in X$ the order of the caustic is defined as the infimum $\kappa(x_0)$ of the numbers κ' such that $u(x, \tau) = O(\tau^{\mu + \kappa'})$ for $\tau \rightarrow \infty$, uniformly for x in a neighborhood of x_0 , for any oscillatory function u of order μ defined by Λ .

Of course $\kappa(x_0) = 0$ for $x_0 \in \pi(\Lambda) \setminus c(\Lambda)$ and $\kappa(x_0) \leq k/2$ where k is the maximum of the dimensions of the intersections $T_{(x_0, \xi)}(\Lambda) \cap T_{(x_0, \xi)}(\text{fiber})$ where $(x_0, \xi) \in \Lambda$.

2.3. Lagrangian distributions. We also need to define caustics of the conic Lagrangian submanifolds. Let X be a C^∞ manifold and $\Lambda \subset T^*(X)$ be a conic C^∞ Lagrangian submanifold. We say that $\phi \in C^\infty(X \times \mathbb{R}^N)$ parametrizes Λ (locally) if

$$(2.6) \quad d_{(x,\theta)} d_\theta \phi \quad \text{has rank } N \text{ when } d_\theta \phi = 0$$

and Λ is locally given by

$$(2.7) \quad \Lambda_\phi = \{(x, d_x \phi(x, \theta)) : d_\theta \phi(x, \theta) = 0\}.$$

Definition 2.4 (Hörmander). Let Λ be a C^∞ conic Lagrangian submanifold in $T^*(X)$. A Lagrangian distribution $u(x)$ of order μ defined by Λ is a locally finite (in X) sum of integrals of the form

$$u(x) = \int e^{i\phi(x, \theta)} a(x, \theta) d\theta,$$

where $\theta \in \mathbb{R}^N$, ϕ satisfies condition (2.6), Λ_ϕ is a piece of Λ and $a(x, \theta) \in S^{\mu - \frac{N}{2} + \frac{\dim X}{4}}(X \times \mathbb{R}^N)$.

We pick a smooth function $\rho \in C_0^\infty([-2, 2])$, $\rho \geq 0$, $\rho|_{[-1, 1]} \equiv 1$. Define $\beta(t) = \rho(t) - \rho(2t)$ for $t > 0$, $\beta \equiv 0$ for $t \leq 0$, so that $\beta \in C_0^\infty([\frac{1}{2}, 2])$. We introduce the Littlewood-Paley decomposition of $u(x)$:

$$(2.8) \quad u_\lambda(x) = \int e^{i\phi(x, \theta)} \beta(|\theta|/\lambda) a(x, \theta) d\theta.$$

Definition 2.5. Let Λ be a conic Lagrangian manifold in $T^*(X)$. The caustic $c(\Lambda)$ of Λ is the projection into X of the set $\Sigma(\Lambda)$ of points in Λ where $\text{rank } d\pi_X < \dim X - 1$. At each point $x_0 \in X$ the order of the caustic is defined as the infimum $\kappa(x_0)$ of the numbers κ' such that $u_\lambda(x) = O(\lambda^{\mu + \frac{\dim X}{4} + \frac{1}{2} + \kappa'})$ for $\lambda \rightarrow \infty$, uniformly for x in a neighborhood of x_0 , for any Lagrangian distribution u of order μ defined by Λ .

Definition 2.6. We say that Λ has a caustic of corank $k \geq 1$ at a point $p_0 \in \Sigma(\Lambda)$ if

$$\text{rank } d\pi_X|_{p_0} = \dim X - 1 - k.$$

Lemma 2.7. Let Λ be a smooth closed conic Lagrangian submanifold of $T^*(X) \setminus 0$. Let $\phi(x, \theta) \in C^\infty(X \times \mathbb{R}^N)$ be a smooth non-degenerate phase function which parametrizes Λ :

$$\Lambda = \{(x, d_x \phi(x, \theta)) : d_\theta \phi(x, \theta) = 0\}.$$

Let $\alpha = \{\alpha_i\}$, $1 \leq i \leq N - 1$, be local coordinates on the unit sphere \mathbb{S}^{N-1} . We use $(\lambda, \alpha) \in \mathbb{R}_+ \times \mathbb{S}^{N-1}$ as local coordinates in \mathbb{R}^N . Then $\mathcal{D} = \det_{ij}(\lambda^{-1} \phi''_{\alpha_i \alpha_j}|_\Lambda)$, $1 \leq i, j \leq N - 1$, is a smooth function on Λ defined up to a nonzero factor:

$$\mathcal{D} \in C^\infty(\Lambda) / C^\infty_\times(\Lambda).$$

This statement is intuitively clear, since \mathcal{D} vanishes precisely on the caustic set $\Sigma(\Lambda)$ where the rank of the projection from Λ onto X is smaller than $\dim X - 1$. Still, since we need to know that the order of vanishing of \mathcal{D} at $\Sigma(\Lambda)$ in particular directions does not depend on the number of oscillatory variables and the choice of local coordinates, we will give a detailed argument in Appendix A.

Definition 2.8. We say that the caustic at a point $p_0 \in \Sigma(\Lambda)$ is simple if it is of corank $k = 1$, so that

$$\text{rank } d\pi_X|_{p_0} = \dim X - 1 - k = \dim X - 2,$$

and if $d_{(x, \theta)} \mathcal{D}|_{p_0} \neq 0$.

Definition 2.9. We say that the simple caustic at a point $p_0 \in \Sigma(\Lambda)$ is of index $m \in \mathbb{N}$ if m is the smallest integer so that there exists a vector field $V \in C^\infty(\Gamma(T(\Lambda)))$, $V|_{\Sigma(\Lambda)} \in \ker d\pi_X$, such that

$$V^m \mathcal{D}(p_0) \neq 0,$$

where

$$\mathcal{D} = \det_{ij}(|\theta|^{-1} \phi''_{\alpha_i \alpha_j}), \quad 1 \leq i, j \leq N - 1.$$

Remark 2.10. Consistency of this definition (independence of the choice of the phase function ϕ that parametrizes Λ and independence of the choice of local coordinates α on \mathbb{S}^{N-1}) follows from Lemma 2.7.

Example 2.11. Let $\theta \in \mathbb{R}^2$ and $\lambda = |\theta|$. Then $\theta/|\theta| \in \mathbb{S}^1$. Denote a local coordinate on \mathbb{S}^1 by α . Consider the phase function $\phi(x, \theta) = |\theta|\Phi(x, \alpha)$, with

$$\Phi(x, \alpha) = \alpha^{m+2} + x_1\alpha^m + \cdots + x_m\alpha + x_{m+1}.$$

This is the model example of a caustic of the type A_{m+1} . The corresponding Lagrangian is $\Lambda = \{x, d\phi(x, \theta) : \phi'_\theta = 0\}$, which can be written as

$$\Lambda = \{x, \lambda d_x \Phi(x, \alpha) : \Phi(x, \alpha) = 0, \Phi'_\alpha(x, \alpha) = 0\}.$$

The Lagrangian could be parametrized by (x', λ, α) , where $x' = (x_1, \dots, x_{m-1})$. At Λ , one can express x_m and x_{m+1} as $x_m = X_m(x', \alpha)$, $x_{m+1} = X_{m+1}(x', \alpha)$.

The kernel of the map $\pi : \Lambda \rightarrow X$ always contains the tangent vector $\partial/\partial\lambda$. On the caustic set

$$c(\Lambda) = \{(x', \alpha) \in \Lambda : \mathcal{D}(x', \alpha) = \Phi''_{\alpha\alpha}(x', x_m(x', \alpha), x_{m+1}(x', \alpha), \alpha) = 0\}$$

one also has $\partial/\partial\alpha \in \ker d\pi$. Since $1 \leq \dim \ker d\pi \leq 2$ and $d\mathcal{D} = d(\Phi''_{\alpha\alpha}|_\Lambda) \neq 0$, the caustics are simple in the sense of Definition 2.8.

Consider the vector field $V = \partial/\partial\alpha \in C^\infty(\Gamma(T\Lambda))$, $V|_{c(\Lambda)} \in \ker d\pi$. Since

$$V^m \mathcal{D}(x', \alpha) = \partial_\alpha^m (\Phi''_{\alpha\alpha}|_\Lambda) = \partial_\alpha^m \Phi''_{\alpha\alpha}(x', x_m(x', \alpha), x_{m+1}(x', \alpha), \alpha) \neq 0,$$

one concludes that the caustic is of index (at most) m .

Remark 2.12. While caustics of the type A_{m+1} , $m \geq 1$, correspond to simple caustics of index m , the converse is not necessarily true, except when $m = 1$ and 2 .

Let us show how to prove that simple caustics of index $m = 1$ and 2 necessarily correspond to caustics of the type A_2 and A_3 , respectively. We first reduce the number of oscillatory variables to $N = 2$, denote by α the local coordinate on \mathbb{S}^1 , and define

$$\Phi(x, \alpha) = \phi(x, \theta/|\theta|) = \phi(x, \theta)/|\theta|.$$

It suffices to notice that ϕ has the caustic of the type A_{m+1} at the point (x_0, α_0) if $\partial_\alpha^j \Phi(x_0, \alpha_0) = 0$, $j < m + 2$, $\partial_\alpha^{m+2} \Phi(x, \alpha) \neq 0$, and if the differentials

$$d\Phi(x, \alpha), d\Phi'_\alpha(x, \alpha), \dots, d\Phi_{\alpha\alpha}^{(m)}(x, \alpha)$$

are linearly independent. In the case $m = 1$, the linear independence of $d\Phi$ and $d\Phi'_\alpha$ follows from the non-degeneracy assumption on ϕ (the differentials $d\phi_{\theta_j}$ are linearly independent).

To settle the case $m = 2$, we additionally need to check that $d\Phi''_{\alpha\alpha}$ is linearly independent of $d\Phi$ and $d\Phi'_\alpha$. We only need to notice that two latter differentials vanish identically on $T\Lambda$, while the differential $d\Phi''_{\alpha\alpha} = d\mathcal{D}$ was assumed to be different from zero (see Definition 2.8).

3. MAIN RESULTS

Consider a Fourier integral operator

$$(3.1) \quad \mathfrak{F}u(x) = \int_{\mathbb{R}^N \times Y} e^{i\phi(x, \theta, y)} a(x, \theta, y) u(y) d\theta dy,$$

where X and Y are two smooth manifolds (unless stated otherwise, we assume that $\dim X = \dim Y = n$), $a(x, \theta, y) \in S_{\text{cl}}^d(X \times \mathbb{R}^N \times Y)$ is a symbol of order d in θ , compactly supported in $X \times Y$. (We restrict the consideration to classical

(polyhomogeneous) symbols, denoted by S_{cl} .) The function ϕ is a non-degenerate phase: the differentials $d_{(x,\theta,y)}\phi'_{\theta_j}$, $1 \leq j \leq N$, are linearly independent, so that

$$\Sigma_\phi = \{(x, \theta, y) : \phi'_\theta(x, \theta, y) = 0\}$$

is a smooth submanifold of $X \times \mathbb{R}^N \times Y$ of dimension $\dim X + \dim Y$. We write $\mathfrak{F} \in I^\mu(X, Y, \mathbf{C})$, where the order of the operator is defined by $\mu = d + \frac{N}{2} - \frac{\dim X + \dim Y}{4}$ and \mathbf{C} is the associated canonical relation:

$$\Sigma_\theta \xrightarrow{\cong} \mathbf{C} \equiv \{(x, d_x \phi(x, \theta, y)), (y, -d_y \phi(x, \theta, y)) : \phi'_\theta(x, \theta, y) = 0\}.$$

According to [Hör71], the minimal number of oscillatory variables is equal to

$$N_{\min} = \dim X + \dim Y - \min \text{rank } d\pi_{X \times Y}.$$

If \mathbf{C} has non-empty caustic set $\Sigma(\mathbf{C})$, then $N_{\min} > 1$.

Definition 3.1. We will say that the canonical relation $\mathbf{C} \subset T^*(X) \setminus 0 \times T^*(Y) \setminus 0$ has a caustic at a point $p_0 = ((x_0, \xi_0), (y_0, \eta_0)) \in \mathbf{C}$ if the twisted canonical relation $\mathbf{C}' = \{((x, \xi), (y, -\eta)) : ((x, \xi), (y, \eta)) \in \mathbf{C}\}$, which is a conic Lagrangian submanifold of $T^*(X \times Y) \setminus 0$, has a caustic at a point $((x_0, \xi_0), (y_0, -\eta_0)) \in \mathbf{C}'$. We will not distinguish the caustics of \mathbf{C} and \mathbf{C}' .

The following result is an immediate consequence of Definitions 2.5 and 3.1:

Theorem 3.2. *Let X, Y be two smooth manifolds (possibly of different dimension), and let $\mathbf{C} \subset T^*(X) \setminus 0 \times T^*(Y) \setminus 0$ be a smooth canonical relation. Let the Fourier integral operator $\mathfrak{F} \in I^\mu(X, Y, \mathbf{C})$ have its symbol compactly supported in $X \times Y$. (The symbol of \mathfrak{F} does not have to be polyhomogeneous.) If \mathbf{C} has caustics of order at most κ , then*

$$(3.2) \quad \mathfrak{F} : L^1(Y) \rightarrow L^\infty(X) \quad \text{if} \quad \mu < -\frac{\dim X + \dim Y}{4} - \frac{1}{2} - \kappa.$$

Further, assume that $\dim X = \dim Y = n$ and that \mathbf{C} is a local graph. Then

$$(3.3) \quad \mathfrak{F} : L^p_{\mu+(n+1+2\kappa)\delta_p}(Y) \rightarrow L^{p'}(X), \quad 1 < p \leq 2, \quad \delta_p = \frac{1}{p} - \frac{1}{2}.$$

Proof. The first part of the theorem follows from the trivial estimate

$$(3.4) \quad \|\mathfrak{F}_\lambda u\|_{L^\infty} \leq C \lambda^{\mu + \frac{\dim X + \dim Y}{4} + \frac{1}{2} + \kappa} \|u\|_{L^1}.$$

For the second part, we interpolate (3.4) with $\|\mathfrak{F}_\lambda\|_{L^2 \rightarrow L^2} \leq C \lambda^\mu$ and apply Littlewood-Paley theory. \square

Definition 3.3. For our convenience, we introduce the map

$$(3.5) \quad (\cdot, \cdot)^\dagger : (p, q) \mapsto (p, q)^\dagger = (1/p, 1/q).$$

Definition 3.4. We define

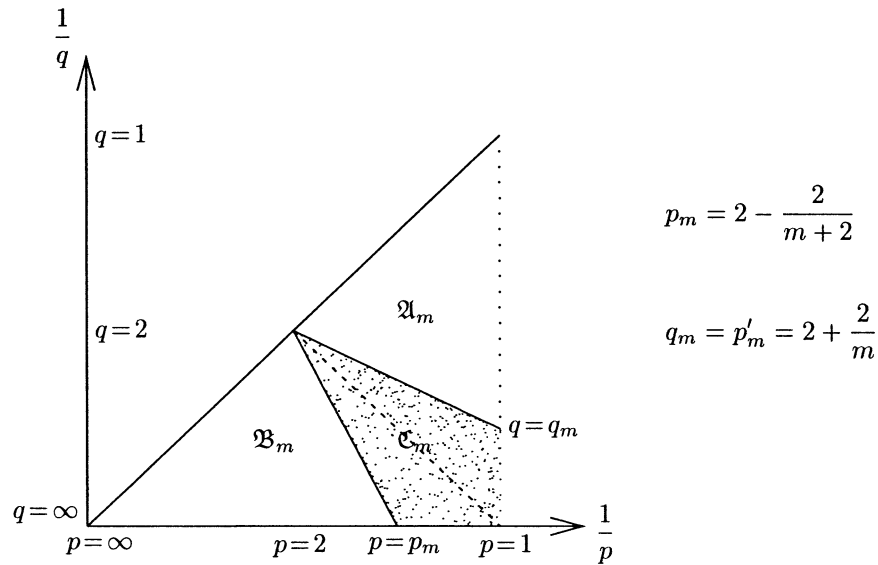
$$p_m = 2 - \frac{2}{m+2}, \quad q_m = p'_m = 2 + \frac{2}{m}$$

and introduce the following regions in $(1/p, 1/q)$ -plane (see Figure 1):

\mathfrak{A}_m is the open triangle with the vertices at $(2, 2)^\dagger$, $(1, 1)^\dagger$, and $(1, q_m)^\dagger$.

\mathfrak{B}_m is the open triangle with the vertices at $(\infty, \infty)^\dagger$, $(2, 2)^\dagger$, and $(p_m, \infty)^\dagger$.

\mathfrak{C}_m is the open convex hull of points $(1, \infty)^\dagger$, $(1, q_m)^\dagger$, $(2, 2)^\dagger$, and $(p_m, \infty)^\dagger$.

FIGURE 1. Regions \mathfrak{A}_m , \mathfrak{B}_m , and \mathfrak{C}_m in $(1/p, 1/q)$ -plane.

The following is our main result:

Theorem 3.5. *Let X, Y be two smooth manifolds, $\dim X = \dim Y = n$, and let $\mathbf{C} \subset T^*(X) \setminus 0 \times T^*(Y) \setminus 0$ be a smooth canonical relation which is a local graph. Assume that \mathbf{C} has only simple caustics of index at most m , $m \in \mathbb{N}$. Let $\mathfrak{F} \in I^\mu(X, Y, \mathbf{C})$ have the classical (polyhomogeneous) symbol with compact support in $X \times Y$. Then for $(p, q)^\dagger \in \mathfrak{A}_m \cup \mathfrak{B}_m$ the $L^p \rightarrow L^q$ estimates on \mathfrak{F} are caustics-insensitive.*

Precisely,

$$(3.6) \quad \mathfrak{F} : L^p_{\mu+n\delta_p+\delta_q}(Y) \rightarrow L^q(X), \quad (p, q)^\dagger \in \mathfrak{A}_m,$$

$$(3.7) \quad \mathfrak{F} : L^p_{\mu+n\delta_q+\delta_p}(Y) \rightarrow L^q(X), \quad (p, q)^\dagger \in \mathfrak{B}_m,$$

where

$$\delta_p = \frac{1}{p} - \frac{1}{2}, \quad \delta_q = \frac{1}{2} - \frac{1}{q}.$$

For $(p, q)^\dagger \in \mathfrak{C}_m$, the estimates depend on the order of the caustic, given by $\kappa = \frac{1}{2} - \frac{1}{m+2}$:

$$(3.8) \quad \mathfrak{F} : L^p_{\mu+n\delta_p+(\delta_p+\delta_q)(1/2+\kappa)+(\delta_q-\delta_p)}(Y) \rightarrow L^q(X), \quad (p, q)^\dagger \in \mathfrak{C}_m, \quad q \leq p',$$

$$(3.9) \quad \mathfrak{F} : L^p_{\mu+n\delta_q+(\delta_p+\delta_q)(1/2+\kappa)+(\delta_p-\delta_q)}(Y) \rightarrow L^q(X), \quad (p, q)^\dagger \in \mathfrak{C}_m, \quad q > p'.$$

Remark 3.6. We restrict the consideration to the class of classical symbols $S_{\text{cl}}^d \subset S_{1,0}^d$ in order to simplify the proof of Lemma 4.3.

Remark 3.7. Note that $\kappa = \frac{1}{2} - \frac{1}{m+2}$ is the order of a caustic of the type A_{m+1} .

Remark 3.8. We need the assumption that \mathbf{C} is a local graph in order to interpolate with the L^2 -based Sobolev estimates on \mathfrak{F} :

$$(3.10) \quad \mathfrak{F} : L^2_\mu(Y) \rightarrow L^2(X).$$

The argument could immediately be adapted to the case when the projections from \mathbf{C} have singularities, as long as $\mathbf{C} \rightarrow X$ and $\mathbf{C} \rightarrow Y$ are assumed to be submersions. In this case, one only needs to modify (3.10), taking into account the loss of derivatives due to singularities of the projections; see [GS02].

Remark 3.9. The sharp estimates on the line $p = q$ follow from [SSS91]:

$$\mathfrak{F} : \mathfrak{h}_{\mu + \frac{n-1}{2}}^1(Y) \rightarrow L^1(X).$$

This map can be interpolated with the continuous $L_\mu^2 \rightarrow L^2$ action. (Generalization for operators with degenerate canonical relations is obtained in [CC03].)

Remark 3.10. On the line segments $((2, 2)^\dagger, (1, q_m)^\dagger)$, $((2, 2)^\dagger, (p_m, \infty)^\dagger)$, and on the lines $p = 1$ and $q = \infty$ the stated estimates hold with the loss of $\epsilon > 0$.

In particular cases, we also have sharp $\mathfrak{h}^1 \rightarrow L^q$ and $L^p \rightarrow \text{BMO}$ estimates, as stated in the next theorem.

Theorem 3.11. *Let X, Y be two smooth manifolds, $\dim X = \dim Y = n$, and let $\mathbf{C} \subset T^*(X) \setminus 0 \times T^*(Y) \setminus 0$ be a smooth canonical relation such that both $\mathbf{C} \rightarrow X$ and $\mathbf{C} \rightarrow Y$ are submersions. Assume that \mathbf{C} has only caustics of the type A_{m+1} with $m = 1$ or 2 . Let $\mathfrak{F} \in I^\mu(X, Y, \mathbf{C})$ have the classical (polyhomogeneous) symbol compactly supported in $X \times Y$. Then*

$$(3.11) \quad \mathfrak{F} : \mathfrak{h}_{\mu + \frac{n}{2} + \delta_q}^1(Y) \rightarrow L^q(X), \quad 2 \leq q < q_m,$$

$$(3.12) \quad \mathfrak{F} : \mathfrak{h}_{\mu + \frac{n}{2} + \delta_q + \kappa_{\frac{\delta_q - \delta_{q_m}}{1/2 - \delta_{q_m}}}}^1(Y) \rightarrow L^q(X), \quad q_m < q \leq \infty.$$

Remark 3.12. In this theorem, we do not need \mathbf{C} to be a local graph.

Remark 3.13. $L^p(Y) \rightarrow \text{BMO}(X)$ estimates on \mathfrak{F} for $1 < p < 2$ are obtained by duality. Other $L^p \rightarrow L^q$ estimates can be obtained by interpolation with L^2 Sobolev estimates.

4. MICROLOCAL TECHNIQUES: DECOMPOSITIONS AND INTERPOLATIONS

In this section, we prove Theorem 3.5.

4.1. Dyadic decompositions. We pick a smooth function $\rho \in C_0^\infty([-2, 2])$, $\rho \geq 0$, $\rho|_{[-1, 1]} \equiv 1$. Define $\beta \in C_0^\infty([\frac{1}{2}, 2])$ by $\beta(t) = \rho(t) - \rho(2t)$ for $t > 0$, $\beta \equiv 0$ for $t \leq 0$. The functions ρ and β define dyadic partition of unity: for any $t \in \mathbb{R}$,

$$\sum_{\pm} \sum_{j \in \mathbb{N}} \beta(\pm 2^{-j} t / 2) + \rho(|t|) = 1.$$

We use the partition of unity which is the Littlewood-Paley decomposition with respect to the magnitude of $|\theta|$ and the dyadic decomposition with respect to the distance \mathcal{D} from $\Sigma(\mathbf{C})$:

$$1 = \left(\sum_{\lambda=2^l, l \in \mathbb{N}} \beta(2^{-l} |\theta|) + \rho(|\theta|) \right) \left(\sum_{\pm} \sum_{j=1}^{j_0-1} \beta(\pm 2^j \mathcal{D}) + \rho(2^{j_0} |\mathcal{D}|) + (1 - \rho(2 |\mathcal{D}|)) \right).$$

We define

$$(4.1) \quad \mathfrak{F}_{\lambda, \pm \sigma} u(x) = \int_{\mathbb{R}^N \times Y} e^{i\phi(x, \theta, y)} \beta(\pm \mathcal{D}(x, \theta, y)/\sigma) \beta(|\theta|/\lambda) a(x, \theta, y) u(y) d\theta dy,$$

$$(4.2) \quad \tilde{\mathfrak{F}}_{\lambda, \sigma} u(x) = \int_{\mathbb{R}^N \times Y} e^{i\phi(x, \theta, y)} \rho(\mathcal{D}(x, \theta, y)/\sigma) \beta(|\theta|/\lambda) a(x, \theta, y) u(y) d\theta dy.$$

We also define

$$(4.3) \quad \mathfrak{F}_{\text{smooth}} u(x) = \int_{\mathbb{R}^N \times Y} e^{i\phi(x, \theta, y)} \rho(|\theta|) a(x, \theta, y) u(y) d\theta dy$$

and

$$(4.4) \quad \mathfrak{F}_{\text{nice}} u(x) = \int_{\mathbb{R}^N \times Y} e^{i\phi(x, \theta, y)} [(1 - \rho(|\theta|))(1 - \rho(2|\mathcal{D}|))] a(x, \theta, y) u(y) d\theta dy.$$

There is a decomposition

$$(4.5) \quad \mathfrak{F} = \sum_{\pm} \sum_{\lambda} \sum_{\sigma > \sigma_0(\lambda)} \mathfrak{F}_{\lambda, \pm \sigma} + \sum_{\lambda} \tilde{\mathfrak{F}}_{\lambda, \sigma_0(\lambda)} + \mathfrak{F}_{\text{smooth}} + \mathfrak{F}_{\text{nice}},$$

where both λ and σ run over powers of 2:

$$\lambda = 2^l, \quad l \in \mathbb{N}, \quad \sigma = 2^{-j}, \quad 1 \leq j < j_0(\lambda) \equiv \lceil \log_2 \lambda^{\frac{m}{m+2}} \rceil.$$

We set $\sigma_0(\lambda) = 2^{-j_0(\lambda)}$, so that

$$(4.6) \quad \sigma_0(\lambda) \approx \lambda^{-\frac{m}{m+2}}.$$

We use the symbol “ \approx ” to indicate that the quantities differ at most by a factor of 2.

The operator $\mathfrak{F}_{\text{smooth}}$ is infinitely smoothing and can be discarded. Since there are no caustics on the support of $1 - \rho(2|\mathcal{D}|)$, the operator $\mathfrak{F}_{\text{nice}}$ can also be discarded. The estimates on operators $\mathfrak{F}_{\lambda, \pm \sigma}$ are the same independent of the sign, and the treatment is the same; we will only consider the “+”-case.

4.2. $L^1 \rightarrow L^\infty$ estimates.

Proposition 4.1. *Let \mathbf{C} and $\mathfrak{F} \in I^\mu(X, Y, \mathbf{C})$ be as in Theorem 3.5, and let $\mathfrak{F}_{\lambda, \sigma}$, $\tilde{\mathfrak{F}}_{\lambda, \sigma}$ be given by (4.1), (4.2). Then*

$$(4.7) \quad \|\mathfrak{F}_{\lambda, \sigma}\|_{L^1 \rightarrow L^\infty} \leq C \lambda^{\mu + \frac{n+1}{2}} \sigma^{-\frac{1}{2}},$$

$$(4.8) \quad \|\mathfrak{F}_{\lambda, \sigma}\|_{L^1 \rightarrow L^\infty} + \|\tilde{\mathfrak{F}}_{\lambda, \sigma}\|_{L^1 \rightarrow L^\infty} \leq C \lambda^{\mu + \frac{n+2}{2}} \sigma^{\frac{1}{m}}.$$

Remark 4.2. The value $\sigma_0(\lambda) \approx \lambda^{-\frac{m}{m+2}}$ in (4.6) is chosen so that the estimates (4.7), (4.8) coincide at $\sigma = \sigma_0(\lambda)$.

Proof. We use the representation of \mathfrak{F} with the minimal possible number of oscillatory variables, $N = 2$. Then $\mathfrak{F}_{\lambda, \sigma}$ could be written as

$$(4.9) \quad \int_{\mathbb{R} \times \mathbb{S}} \int_Y e^{i\phi(x, \tau, \alpha, y)} \rho(\phi''_{\alpha\alpha}(x, 1, \alpha, y)/\sigma) \beta(\tau/\lambda) a(x, \tau, \alpha, y) u(y) \tau d\tau d\alpha dy,$$

where $a(x, \tau, \alpha, y)$ is a classical symbol of order $d = \mu + \frac{n}{2} - 1$. For (4.7), we need the bound

$$(4.10) \quad \left| \int_{\mathbb{R} \times \mathbb{S}} e^{i\phi(x, \tau, \alpha, y)} \rho(\phi''_{\alpha\alpha}(x, 1, \alpha, y)/\sigma) \beta(\tau/\lambda) a(x, \tau, \alpha, y) \tau d\tau d\alpha \right| \leq C \lambda^{\mu + \frac{n+1}{2}} \sigma^{-\frac{1}{2}},$$

uniformly in x and y from a small open neighborhood in $X \times Y$ and for all $\lambda \geq 1$ and $\sigma \leq 1$, $\sigma \geq \lambda^{-\frac{m}{m+2}}$. For simplicity, we assume that $\mu = -n/2$, so that $\tau a(x, \tau, \alpha, y)$ is a symbol of order zero, which we denote by $b(x, \tau, \alpha, y)$. This classical symbol has the development

$$(4.11) \quad b(x, \tau, \alpha, y) \sim b_0(x, \alpha, y) + \sum_{j \in \mathbb{N}} b_j(x, \alpha, y) \tau^{-j}, \quad \tau \geq 1.$$

Denote

$$(4.12) \quad I_{\lambda, \sigma}(x, y) = \lambda^{-1/2} \int_{\mathbb{R}} d\tau \int_K d\alpha e^{i\phi(x, \tau, \alpha, y)} b(x, \tau, \alpha, y) \beta(\tau/\lambda) \beta(\phi''_{\alpha\alpha}(x, 1, \alpha, y)/\sigma),$$

where $K \subset \mathbb{S}$ denotes α -support of $b(x, \tau, \alpha, y)$. Substituting $\tau = \lambda z$, we rewrite $I_{\lambda, \sigma}(x, y)$ as

$$(4.13) \quad I_{\lambda, \sigma}(x, y) = \lambda^{1/2} \int_{1/2}^2 dz \int_K d\alpha e^{i\lambda z \phi(x, 1, \alpha, y)} b(x, \lambda z, \alpha, y) \beta(z) \beta(\phi''_{\alpha\alpha}(x, 1, \alpha, y)/\sigma).$$

For (4.10), we need the bound

$$(4.14) \quad |I_{\lambda, \sigma}(x, y)| \leq C \sigma^{-1/2},$$

valid for all $\lambda \geq 1$, $\lambda^{-m/(m+2)} \leq \sigma \leq 1$, and with C independent of λ and σ .

Lemma 4.3.

$$(4.15) \quad |I_{\lambda, \sigma}(x, y)| \leq C \sigma^{-1/2},$$

with $C < \infty$ independent on $\lambda > 1$ and $0 < \sigma \leq 1$.

We need this estimate to be uniform in λ and σ simultaneously. Similar estimates were considered in [CdV77] and in many other papers. The result is known to be optimal, but is not proved in the whole generality in higher dimensions. For the sake of completeness, we give our own proof for the case we are interested in.

Proof. There is a trivial bound $|I_{\lambda, \sigma}(x, y)| \leq \lambda^{1/2}$, due to the compact support of the integrand. This settles the case $0 < \sigma \leq \lambda^{-1}$; from now on, we assume that $\lambda^{-1} \leq \sigma \leq 1$.

Denote $b'(x, \tau, \alpha, y) = b(x, \tau, \alpha, y) - b_0(x, \alpha, y) \in S_{\text{cl}}^{-1}$, and let

$$(4.16) \quad I'_{\lambda, \sigma}(x, y) = \lambda^{-1/2} \int_{\mathbb{R}} d\tau \int_K d\alpha e^{i\lambda z \phi(x, 1, \alpha, y)} b'(x, \lambda z, \alpha, y) \beta(z) \beta(\phi''_{\alpha\alpha}(x, 1, \alpha, y)/\sigma).$$

Since $|b(x, \lambda z, \alpha, y) \beta(z)| \leq C \lambda^{-1}$, uniformly in $x, \alpha, \lambda \geq 1, 1/2 \leq z \leq 2$, and y , there is an easy bound

$$|I'_{\lambda, \sigma}(x, y)| \leq \lambda^{1/2} \int_{1/2}^2 \int_K |b'(x, \lambda z, \alpha, y)| \beta(z) dz d\alpha \leq C \lambda^{-1/2} \leq C.$$

Thus, we only need to consider the bound on (4.13) with $b_0(x, \alpha, y)$ instead of $b(x, \lambda z, \alpha, y)$:

$$(4.17) \quad I_{\lambda, \sigma}^0(x, y) = \lambda^{1/2} \int_{\mathbb{R}} \int_K e^{i\lambda z \phi(x, 1, \alpha, y)} b_0(x, \lambda z, \alpha, y) \beta(z) \beta(\phi''_{\alpha\alpha}(x, 1, \alpha, y)/\sigma) dz d\alpha.$$

Denoting by $\hat{\beta}$ the Fourier transform of $\beta(z)$, we rewrite $I_{\lambda,\sigma}^0$ as

$$(4.18) \quad I_{\lambda,\sigma}^0(x, y) = \lambda^{1/2} \int_K b_0(x, \alpha, y) \hat{\beta}(\lambda\phi(x, 1, \alpha, y)) \beta(\phi''_{\alpha\alpha}(x, 1, \alpha, y)/\sigma) d\alpha.$$

The statement of the lemma follows from the bound

$$(4.19) \quad |I_{\lambda,\sigma}^0(x, y)| \leq C\sigma^{-1/2},$$

which is uniform in λ , $\lambda^{-1} \leq \sigma < 1$, x , and y . We will prove this bound in the next lemma.

This completes the proof of Lemma 4.3. \square

Lemma 4.4. *Assume that $\phi''_{\alpha\alpha}(\alpha)$ vanishes at most of order m on a compact set $K \subset \mathbb{R}$.*

If $f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $\beta \in C_0^\infty([1/2, 2])$, then

$$(4.20) \quad \lambda^{1/2} \int_K f(\lambda\phi(\alpha)) \beta(\phi''_{\alpha\alpha}(\alpha)/\sigma) d\alpha \leq (\|f\|_{L^1} + \|f\|_{L^\infty}) C\sigma^{-1/2},$$

uniformly in $\lambda > 1$ and $0 < \delta \leq 1$.

Essentially, we are proving the following sublevel set estimate:

$$\left| \{ \alpha \in K : |\phi(\alpha) - \gamma| \leq \lambda^{-1}, \quad |\phi''_{\alpha\alpha}(\alpha)| \geq \sigma^{1/2} \} \right| \leq \frac{C}{\sqrt{\lambda\sigma}} \quad (\text{uniformly in } \gamma \in \mathbb{R}).$$

Proof. The proof of this estimate is simple, so we can give it in detail. Let $\rho \in C_0^\infty([-2, 2])$, $\rho|_{[-1, 1]} \equiv 1$. We use the partition

$$1 = \rho(\phi'_\alpha \sqrt{\lambda/\sigma}) + \left(1 - \rho(\phi'_\alpha \sqrt{\lambda/\sigma})\right)$$

to rewrite (4.20) as a sum of two terms,

$$(4.21) \quad \lambda^{1/2} \int_K f(\lambda\phi) \rho(\phi'_\alpha \sqrt{\lambda/\sigma}) \beta(\phi''_{\alpha\alpha}/\sigma) d\alpha + \lambda^{1/2} \int_K f(\lambda\phi) \left(1 - \rho(\phi'_\alpha \sqrt{\lambda/\sigma})\right) \beta(\phi''_{\alpha\alpha}/\sigma) d\alpha,$$

which we analyze separately.

The first term in (4.21) is bounded by

$$\begin{aligned} & \lambda^{1/2} \|f\|_{L^\infty} \int_K \rho(\phi'_\alpha \sqrt{\lambda/\sigma}) \beta(\phi''_{\alpha\alpha}/\sigma) d\alpha \\ & \leq \lambda^{1/2} \|f\|_{L^\infty} \frac{C}{\sqrt{\lambda/\sigma} \inf |\phi''_{\alpha\alpha}|} \leq C\sigma^{-1/2} \|f\|_{L^\infty}, \end{aligned}$$

since $\inf |\phi''_{\alpha\alpha}| \geq \sigma/2$ on the support of the integrand. The value of C depends on the bound on the number of roots of $\phi'_\alpha(\alpha) = c$ (this number is bounded uniformly in c due to the finite type assumption: $\phi''_{\alpha\alpha}$ vanishes of order at most m).

The second term is bounded by

$$\lambda^{1/2} \int_K f(\lambda\phi(\alpha)) (1 - \rho(\phi'_\alpha \sqrt{\lambda/\sigma})) d\alpha \leq \lambda^{1/2} \|f\|_{L^1} \frac{C}{\lambda \inf |\phi'_\alpha|} \leq C\sigma^{-1/2} \|f\|_{L^1},$$

since $\inf |\phi'_\alpha| \geq \sqrt{\sigma/\lambda}$ on the support of the integrand. Again, we need to mention that the number of roots of $\phi''_{\alpha\alpha}(\alpha) = c$ is bounded uniformly in c due to the finite type assumption.

This proves Lemma 4.4. \square

Remark 4.5. The maximal order of vanishing, $m \in \mathbb{N}$, does not appear in the above lemma. The statement of the lemma is also true without the finite type assumption if we require that ϕ is real analytic, or, more generally, if we require that $\phi \in C^\infty(\mathbb{R})$ and that ϕ'' is “finitely oscillating” on $K \subset \mathbb{R}$:

The number of connected components of the set $\{\alpha \in K : \phi''(\alpha) = c\}$ is bounded uniformly in $c \in \mathbb{R}$.

This assumption holds for any real analytic function, but does not hold for all smooth functions; an example of a smooth function which is “infinitely oscillating” on $[-1, 1]$ is $e^{-1/x^2} \sin(1/x)$.

This finishes the proof of (4.7).

The proof of (4.8) is similar but much more straightforward. One needs to use the following well-known lemma (see, e.g., [CCW99]):

Lemma 4.6. *If $f(\alpha)$ vanishes at most of order m on $[-2, 2]$ and $\beta \in C_0^\infty([-2, 2])$, then $\int_{\mathbb{R}} \beta(f(\alpha)/\sigma) d\alpha$ is bounded by $C\sigma^{1/m}$.*

This finishes the proof of Proposition 4.1. \square

4.3. $L^1 \rightarrow L^2$ estimates.

Lemma 4.7. $\|\mathfrak{F}_{\lambda,\sigma}^* \mathfrak{F}_{\lambda,\sigma}\|_{L^1 \rightarrow L^\infty} + \|\tilde{\mathfrak{F}}_{\lambda,\sigma}^* \tilde{\mathfrak{F}}_{\lambda,\sigma}\|_{L^1 \rightarrow L^\infty} \leq C\lambda^{2\mu+n}\sigma^{\frac{1}{m}}.$

Proof. Since we assume that \mathbf{C} is a local graph (or at least that $\mathbf{C} \rightarrow Y$ is a submersion, as in Theorem 3.11), we can choose the phase function of the form $\phi(x, \theta, y) = x \cdot \theta - S(\theta, y)$, with $\theta \in \mathbb{R}^N$, $N = n$, where $S(\theta, y)$ is homogeneous in θ of degree 1. Then θ and y can be used as the local coordinates on \mathbf{C} . We can rewrite \mathfrak{F} in the form

$$\mathfrak{F}_{\lambda,\sigma} u(x) = \int_{\mathbb{R}^N \times Y} e^{i(x \cdot \theta - S(\theta, y))} a(\theta, y) \beta(\phi(\theta, y)/\sigma) u(y) d\theta dy.$$

$\mathfrak{F}^* \mathfrak{F} \in I^{-2\mu}(X, X, \Delta)$ is a Fourier integral operator with the phase $S(\theta, z) - S(\theta, y)$, associated to the diagonal $\Delta \subset T^*(Y) \times T^*(Y)$, and with $N = n$.

The Fourier integral operator

$$\mathfrak{F}_{\lambda,\sigma}^* \mathfrak{F}_{\lambda,\sigma} u(z) =$$

$$\int_{\mathbb{R}^N \times Y} e^{i(S(\theta, z) - S(\theta, y))} a(\theta, y) \bar{a}(\theta, z) \beta(\mathcal{D}(\theta, y)/\sigma) \beta(\mathcal{D}(\theta, z)/\sigma) \beta^2(|\theta|/\lambda) u(y) d\theta dy$$

has $N = n$ oscillatory variables. (The number of oscillatory variables cannot be reduced since the rank of the matrix $\partial_{\theta_i} \partial_{\theta_j} [S(\theta, z) - S(\theta, y)]$ is zero at $y = z$.) The order of its symbol is 2μ . This yields the bound $\text{const } \lambda^{2\mu+n} \sigma^{\frac{1}{m}}$ on the $L^1 \rightarrow L^\infty$ action, with the factor $\sigma^{\frac{1}{m}}$ due to Lemma 4.6. \square

Remark 4.8. If \mathbf{C} is a local graph, so that $\det_{ij} \partial_{\theta_i} \partial_{y_j} S(\theta, y) \neq 0$, then $\mathfrak{F}^* \mathfrak{F}$ is a pseudodifferential operator.

This lemma yields the following estimate:

Proposition 4.9. *Let \mathbf{C} and $\mathfrak{F} \in I^\mu(X, Y, \mathbf{C})$ be as in Theorem 3.5, and $\mathfrak{F}_{\lambda,\sigma}, \tilde{\mathfrak{F}}_{\lambda,\sigma}$ be given by (4.1), (4.2). Then*

$$(4.22) \quad \|\mathfrak{F}_{\lambda,\sigma}\|_{L^1 \rightarrow L^2} + \|\tilde{\mathfrak{F}}_{\lambda,\sigma}\|_{L^1 \rightarrow L^2} \leq C\lambda^{\mu+\frac{n}{2}} \sigma^{\frac{1}{2m}}.$$

4.4. $L^p \rightarrow L^q$ estimates for $1 < p \leq 2 \leq q < \infty$. Even when we cannot prove the sharp $\mathfrak{h}^1 \rightarrow L^q$ estimates (without the loss of $\epsilon > 0$), we still can prove the sharp $L^p \rightarrow L^q$ estimates, for certain values of p and q which also satisfy $1 < p \leq 2$, $2 \leq q < \infty$. The main tool is the Littlewood-Paley theory.

We group the pieces $\mathfrak{F}_{\lambda,\sigma}$ and $\tilde{\mathfrak{F}}_{\lambda,\sigma}$ defined by (4.1), (4.2) into λ -clusters:

$$(4.23) \quad \mathfrak{F}_\lambda = \sum_{\sigma=2^{-j}, j \in \mathbb{N}, \sigma > \sigma_0(\lambda)} F_{\lambda,\sigma} + \tilde{\mathfrak{F}}_{\lambda,\sigma_0(\lambda)}.$$

Let us consider this series in the norm of operators from $L^1_{\mu+\frac{n}{2}+\delta_q}(Y)$ to $L^q(X)$.

Proposition 4.10. *Let \mathbf{C} and $\mathfrak{F} \in I^\mu(X, Y, \mathbf{C})$ be as in Theorem 3.5, and let \mathfrak{F}_λ be given by (4.23). Then*

$$(4.24) \quad \|\mathfrak{F}_\lambda\|_{L^1 \rightarrow L^q} \leq C \lambda^{\mu+\frac{n}{2}+\delta_q}, \quad 2 \leq q < q_m,$$

$$(4.25) \quad \|\mathfrak{F}_\lambda\|_{L^1 \rightarrow L^q} \leq C \lambda^{\mu+\frac{n}{2}+\delta_q+\kappa \frac{\delta_q-\delta_{q_m}}{1/2-\delta_{q_m}}}, \quad q_m < q < \infty.$$

Proof. Interpolating $L^1 \rightarrow L^\infty$ estimates from Proposition 4.1 with $L^1 \rightarrow L^2$ estimates from Proposition 4.9, we obtain:

Lemma 4.11.

$$(4.26) \quad \|\mathfrak{F}_{\lambda,\sigma}\|_{L^1 \rightarrow L^q} \leq C \lambda^{\mu+\frac{n}{2}+\delta_q} \sigma^{\frac{1}{2m} - (\frac{1}{m}+1)\delta_q}.$$

Therefore, (4.23) is dominated by the geometric series.

If $\frac{1}{2m} > (\frac{1}{m}+1)\delta_q$ (equivalent with $q < q_m$), then the geometric series is convergent, and hence is bounded uniformly in λ . This proves (4.24).

If $\frac{1}{2m} < (\frac{1}{m}+1)\delta_q$ (equivalent with $q > q_m$), the series (4.23) is dominated by finitely many terms of the divergent geometric series:

$$(4.27) \quad \begin{aligned} \|\mathfrak{F}_\lambda\|_{L^1 \rightarrow L^q} &\leq C \sum_{\sigma=2^{-j}, j \in \mathbb{N}, \sigma \geq \sigma_0(\lambda)} \lambda^{\mu+\frac{n}{2}+\delta_q} \sigma^{-((\frac{1}{m}+1)\delta_q - \frac{1}{2m})} \\ &\leq C \lambda^{\mu+\frac{n}{2}+\delta_q} \sigma_0(\lambda)^{-((\frac{1}{m}+1)\delta_q - \frac{1}{2m})}. \end{aligned}$$

Taking into account that $\sigma_0(\lambda) = \lambda^{-\frac{m}{m+2}}$ and that

$$(4.28) \quad \begin{aligned} \frac{m}{m+2} \left(\left(\frac{1}{m} + 1 \right) \delta_q - \frac{1}{2m} \right) &= \frac{1}{2(m+2)} (2(1+m)\delta_q - 1) \\ &= \frac{\kappa}{m} (\delta_q / \delta_{q_m} - 1) = \kappa \frac{\delta_q - \delta_{q_m}}{1/2 - \delta_{q_m}}, \end{aligned}$$

we can rewrite (4.27) in a more convenient form:

$$(4.29) \quad \|\mathfrak{F}_\lambda\|_{L^1 \rightarrow L^q} \leq C \lambda^{\mu+\frac{n}{2}+\delta_q+\kappa \frac{\delta_q-\delta_{q_m}}{1/2-\delta_{q_m}}}.$$

This proves (4.25). \square

The estimates stated in Proposition 4.10 can be interpolated with the $L^2 \rightarrow L^2$ -estimates. If we assume that \mathbf{C} is a local graph, then $\mathfrak{F}_\lambda : L^2_\mu \rightarrow L^2$, and we obtain, for $1 \leq p \leq 2$:

$$\begin{aligned} \|\mathfrak{F}_\lambda\|_{L^p \rightarrow L^q} &\leq C \lambda^{\mu+n\delta_p+\delta_q}, \quad 2 \leq q < \frac{2}{1-4\delta_p\delta_{q_m}}, \\ \|\mathfrak{F}_\lambda\|_{L^p \rightarrow L^q} &\leq C \lambda^{\mu+n\delta_p+(\delta_p+\delta_q)(1/2+\kappa)+(\delta_q-\delta_p)}, \quad \frac{2}{1-4\delta_p\delta_{q_m}} < q < p'. \end{aligned}$$

According to Littlewood-Paley theory ([See93], Lemma 2.1), $\mathfrak{F} = \sum_{\lambda=2^l, l \in \mathbb{N}} \mathfrak{F}_\lambda$ has the same $L^p \rightarrow L^q$ regularity properties as long as $1 < p \leq 2 \leq q < \infty$. This, together with the duality arguments, proves Theorem 3.5.

5. MICROLOCAL TECHNIQUES: $\mathfrak{h}^1 \rightarrow L^q$ ESTIMATES

We are going to prove Theorem 3.11, which gives the substitute of the $L^p \rightarrow L^q$ estimates for $p = 1$ ($\mathfrak{h}^1 \rightarrow L^q$ estimates) and for $q = \infty$ ($L^p \rightarrow \text{BMO}$ estimates).

5.1. $\mathfrak{h}^1 \rightarrow L^\infty$ estimates. The following is the analogue of Proposition 4.1.

Proposition 5.1. *Let $\mathbf{C} \subset T^*(X) \setminus 0 \times T^*(Y) \setminus 0$ be a smooth canonical relation such that $\mathbf{C} \rightarrow X$ is a submersion. Assume that \mathbf{C} has only caustics of the type A_{m+1} with $m = 1$ or 2 . Let $\mathfrak{F} \in I^\mu(X, Y, \mathbf{C})$ have the polyhomogeneous symbol with compact support in X, Y , and let $\mathfrak{F}_{\lambda, \sigma}, \tilde{\mathfrak{F}}_{\lambda, \sigma}$ be given by (4.1), (4.2). Then, for any atom a_Q supported in the cube Q with side r , we have*

$$(5.1) \quad \|\mathfrak{F}_{\lambda, \sigma} a_Q\|_{L^\infty} \leq C \lambda^{\mu + \frac{n+1}{2}} \sigma^{-\frac{1}{2}} \min(\lambda r, (\lambda r)^{-1}),$$

$$(5.2) \quad \|\mathfrak{F}_{\lambda, \sigma} a_Q\|_{L^\infty} + \|\tilde{\mathfrak{F}}_{\lambda, \sigma} a_Q\|_{L^\infty} \leq C \lambda^{\mu + \frac{n+2}{2}} \sigma^{\frac{1}{m}} \min(\lambda r, (\lambda r)^{-1}).$$

Proof. The proof is similar to [SSS91], [CC03]. For the reader's convenience, we reproduce this proof in Appendix B. We require that $\sigma \geq \lambda^{-1/2}$ (equivalent to $m \leq 2$) so that the localizations would not be too fine and the integration by parts from [SSS91] could be used verbatim. \square

We group the pieces $\mathfrak{F}_{\lambda, \sigma}$ into λ -clusters as in (4.23):

$$\mathfrak{F}_\lambda = \sum_{\sigma=2^{-j}, j \in \mathbb{N}, \sigma > \sigma_0(\lambda)} \mathfrak{F}_{\lambda, \sigma} + \tilde{\mathfrak{F}}_{\lambda, \sigma_0(\lambda)}.$$

The estimates (5.1) and (5.2) yield the following bounds on $\|\mathfrak{F}_\lambda a_Q\|_{L^\infty}$:

Corollary 5.2. *Assume that $\mathbf{C} \rightarrow X$ is a submersion and that \mathbf{C} has only caustics of the type A_{m+1} with $m = 1$ or 2 . Then*

$$(5.3) \quad \|\mathfrak{F}_\lambda a_Q\|_{L^\infty} \leq C \lambda^{\mu + \frac{n+1}{2} + \kappa} \min(\lambda r, (\lambda r)^{-1}), \quad \kappa = \frac{1}{2} - \frac{1}{m+2}.$$

This allows us to conclude that

$$(5.4) \quad \mathfrak{F} : \mathfrak{h}_{\mu + \frac{n+1}{2} + \kappa}^1(Y) \rightarrow L^\infty(X).$$

5.2. $\mathfrak{h}^1 \rightarrow L^2$ estimates. Proposition 5.1 gives the sharp version of Proposition 4.1. Now we are going to prove the sharp version of Proposition 4.9.

Lemma 5.3. *Assume that $\mathbf{C} \rightarrow Y$ is a submersion and that \mathbf{C} has only caustics of the type A_{m+1} with $m = 1$ or 2 . Then, for any atom a_Q supported in the cube Q with side r , we have*

$$(5.5) \quad \|\mathfrak{F}_{\lambda, \sigma} \mathfrak{F}_{\lambda, \sigma} a_Q\|_{L^\infty} + \|\tilde{\mathfrak{F}}_{\lambda, \sigma}^* \tilde{\mathfrak{F}}_{\lambda, \sigma} a_Q\|_{L^\infty} \leq C \lambda^{2\mu + n} \sigma^{\frac{1}{m}} \min(\lambda r, (\lambda r)^{-1}).$$

Proof. The proof is similar to the proof of Proposition 4.9. For the $\mathfrak{h}^1 \rightarrow L^\infty$ estimates, we can apply the usual machinery as long as $\min \sigma \approx \lambda^{-\frac{m}{m+2}}$ is not smaller than $\lambda^{-1/2}$, that is, as long as $m \leq 2$. \square

This lemma proves the following sharp version of Proposition 4.9.

Proposition 5.4. *Let $\mathbf{C} \subset T^*(X) \setminus 0 \times T^*(Y) \setminus 0$ be a smooth canonical relation such that $\mathbf{C} \rightarrow Y$ is a submersion. Assume that \mathbf{C} has only caustics of the type A_{m+1} with $m = 1$ or 2 . Let $\mathfrak{F} \in I^\mu(X, Y, \mathbf{C})$ have the polyhomogeneous symbol with compact support in X, Y , and let $\mathfrak{F}_{\lambda, \sigma}, \tilde{\mathfrak{F}}_{\lambda, \sigma}$ be given by (4.1), 4.2. Then, for any atom a_Q supported in the cube Q with side r , we have*

$$(5.6) \quad \|\mathfrak{F}_{\lambda, \sigma} a_Q\|_{L^2} + \|\tilde{\mathfrak{F}}_{\lambda, \sigma} a_Q\|_{L^2} \leq C \lambda^{\mu + \frac{n}{2}} \sigma^{\frac{1}{2m}} \min((\lambda r)^{1/2}, (\lambda r)^{-1/2}).$$

5.3. $\mathfrak{h}^1 \rightarrow L^q$ estimates for small $2 \leq q < q_m$: σ -interpolation. We group the pieces $\mathfrak{F}_{\lambda, \sigma}$ and $\tilde{\mathfrak{F}}_{\lambda, \sigma}$ into σ -clusters:

$$(5.7) \quad \mathfrak{F}_\sigma = \sum_{\substack{\lambda: \sigma \geq 2\sigma_0(\lambda) \\ \lambda=2^l, l \in \mathbb{N}}} \mathfrak{F}_{\lambda, \sigma} + \sum_{\substack{\lambda: \sigma_0(\lambda) \leq \sigma < 2\sigma_0(\lambda) \\ \lambda=2^l, l \in \mathbb{N}}} \tilde{\mathfrak{F}}_{\lambda, \sigma}.$$

Then we have

$$\mathfrak{F} = \sum_{\sigma=2^{-j}, j \in \mathbb{N}} \mathfrak{F}_\sigma + \mathfrak{F}_{\text{nice}}.$$

Proposition 5.1 proves the following bound:

Lemma 5.5. *Assume that $\mathbf{C} \rightarrow X$ is a submersion and that \mathbf{C} has only caustics of the type A_{m+1} with $m = 1$ or 2 . Let \mathfrak{F}_σ be given by (5.7). Then*

$$(5.8) \quad \|\mathfrak{F}_\sigma\|_{\mathfrak{h}^1_{\mu + \frac{n+1}{2}} \rightarrow L^\infty} \leq C \sigma^{-1/2}.$$

Proposition 5.4 proves the following:

Lemma 5.6. *Assume that $\mathbf{C} \rightarrow Y$ is a submersion and that \mathbf{C} has only caustics of the type A_{m+1} with $m = 1$ or 2 . Let \mathfrak{F}_σ be given by (5.7). Then*

$$(5.9) \quad \|\mathfrak{F}_\sigma\|_{\mathfrak{h}^1_{\mu + \frac{n}{2}} \rightarrow L^2} \leq C \sigma^{\frac{1}{2m}}.$$

Corollary 5.7. *Assume that both $\mathbf{C} \rightarrow X$ and $\mathbf{C} \rightarrow Y$ are submersions and that \mathbf{C} has only caustics of the type A_{m+1} with $m = 1$ or 2 . Let \mathfrak{F}_σ be given by (5.7). Then the interpolation of (5.8) and (5.9) gives*

$$(5.10) \quad \|\mathfrak{F}_\sigma\|_{\mathfrak{h}^1_{\mu + \frac{n}{2} + \delta_q} \rightarrow L^q} \leq C \sigma^{\frac{1}{2m} - (\frac{1}{m} + 1)\delta_q}, \quad 2 \leq q \leq \infty.$$

The summation $\sum_{\sigma=2^{-j}, j \in \mathbb{N}} \mathfrak{F}_\sigma$ converges in $\mathfrak{h}^1_{\mu + \frac{n}{2} + \delta_q} \rightarrow L^q$ operator norm (where $q \geq 2$) if

$$\frac{1}{2m} > \left(\frac{1}{m} + 1\right) \delta_q,$$

which is equivalent to $2 \leq q < q_m$, $q_m = 2 + \frac{2}{m}$. In this case, we conclude that

$$(5.11) \quad \mathfrak{F} : \mathfrak{h}^1_{\mu + \frac{n}{2} + \delta_q} \rightarrow L^q, \quad 2 \leq q < q_m.$$

Note that the estimates (5.11) do not depend on the order of caustics.

5.4. $\mathfrak{h}^1 \rightarrow L^q$ estimates for $q > q_m$: ω -interpolation. In the case $m \leq 2$, we can derive the sharp $\mathfrak{h}^1 \rightarrow L^q$ estimates for $q > q_m$. According to Proposition 5.1, if $m \leq 2$ and if a_Q is an atom supported in the cube Q with side r , then

$$(5.12) \quad \|\mathfrak{F}_{\lambda,\sigma} a_Q\|_{L^\infty} + \|\tilde{\mathfrak{F}}_{\lambda,\sigma} a_Q\|_{L^\infty} \leq C \lambda^{\mu+\frac{n+2}{2}} \sigma^{\frac{1}{m}} \min(\lambda r, (\lambda r)^{-1}),$$

$$(5.13) \quad \|\mathfrak{F}_{\lambda,\sigma} a_Q\|_{L^\infty} \leq C \lambda^{\mu+\frac{n+1}{2}+\kappa} \lambda^{-\kappa} \sigma^{-\frac{1}{2}} \min(\lambda r, (\lambda r)^{-1}).$$

According to Corollary 5.4,

$$(5.14) \quad \|\mathfrak{F}_{\lambda,\sigma} a_Q\|_{L^2} + \|\tilde{\mathfrak{F}}_{\lambda,\sigma} a_Q\|_{L^2} \leq C \lambda^{\mu+\frac{n}{2}-\frac{\kappa}{m}} \lambda^{\frac{\kappa}{m}} \sigma^{\frac{1}{2m}} \min((\lambda r)^{1/2}, (\lambda r)^{-1/2}).$$

We introduce a new parameter, ω , for the values of $\lambda^{-\kappa} \sigma^{-\frac{1}{2}}$ (these values are bounded by 1 since $\sigma \geq \sigma_0(\lambda) \approx \lambda^{-2\kappa}$). Let us group the operators $\mathfrak{F}_{\lambda,\sigma}$ into ω -clusters \mathfrak{F}_ω , $\omega = 2^{-k}$, $k \in \mathbb{N}$, so that

$$(5.15) \quad \mathfrak{F} = \mathfrak{F}_{\text{nice}} + \sum_{\omega=2^{-k}, k \in \mathbb{N}} \mathfrak{F}_\omega,$$

where

$$(5.16) \quad \mathfrak{F}_\omega = \sum_{\substack{\omega \leq \lambda^{-\kappa} \sigma^{-1/2} < 2\omega \\ \sigma \geq 2\sigma_0(\lambda)}} \mathfrak{F}_{\lambda,\sigma} + \sum_{\substack{\omega \leq \lambda^{-\kappa} \sigma^{-1/2} < 2\omega \\ \sigma_0(\lambda) \leq \sigma < 2\sigma_0(\lambda)}} \tilde{\mathfrak{F}}_{\lambda,\sigma_0(\lambda)},$$

$\kappa = \frac{1}{2} - \frac{1}{m+2}$, $\lambda = 2^l$, $l \in \mathbb{N}$ and $\sigma = 2^{-j}$, $j \in \mathbb{N}$.

Lemma 5.8. *Assume that both $\mathbf{C} \rightarrow X$ and $\mathbf{C} \rightarrow Y$ are submersions and that \mathbf{C} has only caustics of the type A_{m+1} with $m = 1$ or 2 . Let \mathfrak{F}_ω be given by (5.16). Then, for any atom a_Q supported in the cube Q with side r , we have*

$$(5.17) \quad \|\mathfrak{F}_\omega\|_{\mathfrak{h}^1_{\mu+\frac{n+1}{2}+\kappa} \rightarrow L^\infty} \leq C\omega,$$

$$(5.18) \quad \|\mathfrak{F}_\omega\|_{\mathfrak{h}^1_{\mu+\frac{n}{2}-\frac{\kappa}{m}} \rightarrow L^2} \leq C\omega^{-1/m}.$$

Corollary 5.9. *Assume that both $\mathbf{C} \rightarrow X$ and $\mathbf{C} \rightarrow Y$ are submersions and that \mathbf{C} has only caustics of the type A_{m+1} with $m = 1$ or 2 . Let \mathfrak{F}_ω be given by (5.16). Then*

$$(5.19) \quad \|\mathfrak{F}_\omega\|_{\mathfrak{h}^1_{\mu+\frac{n}{2}-\frac{\kappa}{m}+\delta_q \frac{2m+3}{m+2}} \rightarrow L^q} \leq C\omega^{-\frac{1}{m}+2(\frac{1}{m}+1)\delta_q}.$$

The series $\sum_{\omega=2^{-j}, j \in \mathbb{N}} \mathfrak{F}_\omega$ (considered in $\mathfrak{h}^1_{\mu+\frac{n}{2}-\frac{\kappa}{m}} \rightarrow L^q$ operator norm) is dominated by the geometric series which is convergent if $\frac{1}{m} < 2(\frac{1}{m}+1)\delta_q$, which is equivalent with $q > q_m$. Therefore,

$$(5.20) \quad \mathfrak{F} : \mathfrak{h}^1_{\mu+\frac{n}{2}+\delta_q+\kappa \frac{\delta_q-\delta_{q_m}}{1/2-\delta_{q_m}}} (Y) \rightarrow L^q(X), \quad q > q_m.$$

This finishes the proof of Theorem 3.11.

6. ESTIMATES FOR THE HALF-WAVE OPERATOR

Let (M, g) be a compact Riemann manifold of dimension n . Let $P = \sqrt{-\Delta + 1}$, where Δ is the Laplace operator. The principal symbol $p(x, \xi) = g^{ij}(x)\xi_i\xi_j$ of P generates the Hamiltonian flow $\Phi_t : T^*M \rightarrow T^*M$; this flow leaves invariant the cosphere bundle

$$S^*M = \{(x, \xi) \in T^*M : p(x, \xi) = 1\}.$$

The geodesics of unit speed on M are the curves $t \mapsto \pi\Phi_t((x, \xi))$, $(x, \xi) \in S^*M$. Let π be the canonical projection $T^*M \rightarrow M$. We say that the time t is non-conjugate if the bicharacteristics which start at the moment $t = 0$ at any point $x \in M$ do not form caustics in time t , so that $\pi\Phi_t : S^*M \rightarrow M$ is of maximal rank:

$$(6.1) \quad \text{rank } d\left(\pi\Phi_t|_{S_x^*M}\right)(\xi) = n - 1.$$

Here ξ is a point in the fiber S_x^*M of the cosphere bundle at the point x .

Assume that at $t = T$ the map $\pi\Phi_t : S^*M \rightarrow M$ is no longer of maximal rank at the point (x, ξ) , where $\xi \in S_x^*M$:

$$(6.2) \quad \text{rank } d\left(\pi\Phi_t|_{S_x^*M}\right)(\xi) < n - 1, \quad t = T.$$

The integral kernel K^t of the half-wave operator e^{itP} can be represented as a finite sum of oscillatory integrals of the form

$$K^t(x, y) = \int_{\mathbb{R}^n} e^{i(x \cdot \xi - \phi(t, y, \xi))} a_t(y, \xi) d\xi,$$

where $a_t(y, \xi)$ is a classical symbol of order 0. (See [Sog93], Section 4.) This representation is valid for (x, ξ, y) supported in a small open conic neighborhood of $M \times \mathbb{R}^n \times M$ and t in a small open neighborhood of T . We apply our results on $L^p \rightarrow L^q$ estimates (Theorem 3.5) to the half-wave operator e^{itP} with the integral kernel $K^t(x, y)$.

Theorem 6.1. *If for $t \leq T$ the geodesic flow Φ_t forms only caustics of the type $A_{m'+1}$ with $m' \leq m$, then for $0 < t \leq T$ and for $1 < p \leq p < \infty$ such that $(p, q)^\dagger \notin \mathfrak{C}_m$ the $L^p \rightarrow L^q$ estimates are caustics-insensitive.*

Precisely,

$$(6.3) \quad e^{itP} P^{-n\delta_p - \delta_q} : L^p \rightarrow L^q, \quad (p, q)^\dagger \in \mathfrak{A}_m,$$

$$(6.4) \quad e^{itP} P^{-n\delta_q - \delta_p} : L^p \rightarrow L^q, \quad (p, q)^\dagger \in \mathfrak{B}_m.$$

For $(p, q)^\dagger \in \mathfrak{C}_m$, the estimates depend on the order of the caustic, which is given by $\kappa = \frac{1}{2} - \frac{1}{m+2}$:

$$(6.5) \quad e^{itP} P^{-n\delta_p - (\delta_p + \delta_q)(1/2 + \kappa) - (\delta_q - \delta_p)} : L^p \rightarrow L^q, \quad (p, q)^\dagger \in \mathfrak{C}_m, \quad q \leq p',$$

$$(6.6) \quad e^{itP} P^{-n\delta_q - (\delta_p + \delta_q)(1/2 + \kappa) - (\delta_p - \delta_q)} : L^p \rightarrow L^q, \quad (p, q)^\dagger \in \mathfrak{C}_m, \quad q > p'.$$

The regions \mathfrak{A}_m , \mathfrak{B}_m , and \mathfrak{C}_m in $(1/p, 1/q)$ -plane are defined in Definition 3.4 (see also Figure 1).

We can use these results to investigate precisely the blow-up of the solution just before the formation of the caustics. At non-conjugate times t , the estimates on the half-wave operator e^{itP} are given by the estimates (6.5), (6.6) with $\kappa = 0$. As t approaches the moment T when the geodesic flow starts forming caustics, these estimates blow up (and the estimates with nonzero κ are to be used). As was shown in [Mag01], if $T > 0$ is such that t is non-conjugate for $t \in (T - \epsilon, T)$, for some $\epsilon > 0$, then the $L^{q'} \rightarrow L^q$ estimates on the half-wave operator e^{itP} may blow up as $t \rightarrow T$ at most as

$$(6.7) \quad \|e^{itP} P^{-(n+1)/2}\|_{L^{q'} \rightarrow L^q} \leq C_{q,M}(T) |T - t|^{-K(n-1)\delta_q}, \quad 2 \leq q < \infty,$$

where $K = 4$. This is an a priori value; K could be shown to be smaller when the geodesic flow forms some particular caustics.

Theorem 6.2. *Let $T > 0$ and suppose there exists $\epsilon > 0$ such that t is non-conjugate for $T - \epsilon \leq t < T$. Assume that for $T \leq t \leq T + \epsilon$ the geodesic flow Φ_t forms only simple caustics of index at most m (e.g. caustics of the type A_{m+1}). Let $2 \leq q < \infty$, $1/q + 1/q' = 1$. We have for $T - \epsilon/2 \leq t < T$:*

$$(6.8) \quad \|e^{itP} P^{-(n+1)\delta_q}\|_{L^{q'}(M) \rightarrow L^q(M)} \leq C_{q,M}(T) |T - t|^{-\delta_q}, \quad 2 \leq q < \infty,$$

$$(6.9) \quad \|e^{itP} P^{-(n+1)\delta_q - 2\kappa\delta_q}\|_{L^{q'}(M) \rightarrow L^q(M)} \leq C_{q,M}(T), \quad 2 \leq q < \infty,$$

where $\delta_q = \frac{1}{2} - \frac{1}{q}$ and $\kappa = \frac{1}{2} - \frac{1}{m+2}$.

Proof. We reduce the number of oscillatory variables in the representation of K^t to 2, which is possible in an open neighborhood of simple caustics, and use the polar coordinates $(\lambda, \alpha) \in \mathbb{R}_+ \times \mathbb{S}$ in the θ -space. We exploit the fact that $|\phi''_{\alpha\alpha}| \geq \text{const} |T - t|$ if t is non-conjugate for $T - \epsilon < t < T$. (This bound is easy for stable caustics. For the generic situation, see [Mag01], Lemma 2.4.) We also use the Littlewood-Paley decomposition for K^t (to interpolate $L^1 \rightarrow L^\infty$ estimates on K_λ^t with $L^2 \rightarrow L^2$ estimates). The rest of the theorem is the same as the proof of the statement (4.7) of Proposition 4.1. Again, the optimal estimate with the factor $|\det \phi''_{\alpha\alpha}|^{-1/2}$ for the oscillatory integral is readily available since α is one-dimensional. \square

The interpolation of the $L^p \rightarrow L^q$ estimates which remain valid at the caustics (Theorem 6.1) and the asymptotics which describe the blow-up of the usual $L^p \rightarrow L^{p'}$ estimates (Theorem 6.2) gives the complete description of the behavior of the blow-up of $L^p \rightarrow L^q$ estimates just before the geodesic flow forms caustics.

APPENDIX A. CONSISTENCY OF THE DEFINITION OF \mathcal{D}

In this section we prove Lemma 2.7:

Lemma A.1 (Lemma 2.7). *Let Λ be a smooth closed conic Lagrangian submanifold of $T^*(X) \setminus 0$. Let $\phi(x, \theta) \in C^\infty(X \times \mathbb{R}^N)$ be a smooth non-degenerate phase function which parametrizes Λ :*

$$\Lambda = \{(x, d_x \phi(x, \theta)) : d_\theta \phi(x, \theta) = 0\}.$$

Let $\alpha = \{\alpha_i\}$, $1 \leq i \leq N - 1$, be local coordinates on the unit sphere \mathbb{S}^{N-1} . We use $(\lambda, \alpha) \in \mathbb{R}_+ \times \mathbb{S}^{N-1}$ as local coordinates in \mathbb{R}^N . Then $\mathcal{D} = \det_{ij}(\lambda^{-1} \phi''_{\alpha_i \alpha_j}|_\Lambda)$, $1 \leq i, j \leq N - 1$, is a smooth function on Λ defined up to a nonzero factor:

$$\mathcal{D} \in C^\infty(\Lambda) / C^\infty_\times(\Lambda).$$

We split the proof into two parts: In the first part, we will show that if we use the maximal number of oscillatory variables, then \mathcal{D} is defined up to a nonzero factor. In the second part, we show that \mathcal{D} is multiplied by a nonzero factor if we reduce the number of oscillatory variables.

(i) Let us check that, up to a factor, \mathcal{D} does not depend on the chosen parametrization of Λ if we use the maximal number $N = n$ of oscillatory variables. Λ can be parametrized (locally) by θ with $\theta \in \mathbb{R}^N$, $N = n$. Assume there are two different phase functions $\phi(x, \theta)$ and $\psi(x, \vartheta)$, $\theta \in \mathbb{R}^N$, $\vartheta \in \mathbb{R}^N$, and that both θ and ϑ can be used as local coordinates on Λ . According to e.g. [Dui96], there is a function

$g(x, \theta)$, homogeneous of degree 1 in θ , such that $\phi(x, \theta) = \psi(x, g(x, \theta))$. We rewrite ϕ and ψ as

$$(A.1) \quad \phi = \phi(x, \lambda, \alpha), \quad \psi = \psi(x, \tau, \beta),$$

where $\lambda = |\theta|$, $\tau = |\theta|$, and α, β are local coordinates on \mathbb{S}^{N-1} . Then there is a smooth function $\beta(x, \alpha)$ and a smooth function $c(x, \alpha) \neq 0$ such that

$$(A.2) \quad \phi(x, \lambda, \alpha) = \psi(x, c(x, \alpha)\lambda, \beta(x, \alpha)).$$

To simplify the notations, we will assume the summation with respect to the repeating indices and will not write the subscripts of α, β, \dots at all, assuming that e.g. $\det A''_{\alpha\alpha'}$ stands for $\det_{ij} \partial_{\alpha_i} \partial_{\alpha_j} A$ and $A''_{\alpha\alpha'} d\alpha d\alpha'$ stands for $\sum_{ij} \partial_{\alpha_i} \partial_{\alpha_j} A d\alpha_i d\alpha_j$.

We differentiate relation (A.2) twice with respect to α :

$$\phi''_{\alpha\alpha'} = \psi''_{\beta\beta'} J_{\alpha}^{\beta} J_{\alpha'}^{\beta'} + \psi'_{\beta} J_{\alpha\alpha'}^{\beta} + \psi''_{\beta\lambda} J_{\alpha}^{\beta} c_{\alpha'} + \psi'_{\beta\lambda} J_{\alpha'}^{\beta} c_{\alpha} + \psi''_{\lambda\lambda} c_{\alpha}^2 + \psi'_{\lambda} c_{\alpha\alpha'},$$

where $J_{\alpha}^{\beta}(x, \alpha) = \partial\beta(x, \alpha)/\partial\alpha$ (that is, $J_{\alpha_i}^{\beta_j}(x, \alpha) = \partial\beta_j(x, \alpha)/\partial\alpha_i$). Taking into account that $\psi''_{\lambda\lambda} \equiv 0$ (ψ is homogeneous of degree 1 in λ), while ψ'_{λ} , ψ'_{β} , and $\psi''_{\lambda\beta}(x, \lambda, \beta) = \lambda^{-1}\psi'_{\beta}(x, \lambda, \beta)$ vanish identically on Λ , we deduce that

$$\det \phi''_{\alpha\alpha'}(x, \lambda, \alpha) = (\det J_{\alpha}^{\beta}(x, \alpha))^2 \det \psi''_{\beta\beta'}(x, c(x, \alpha)\lambda, \beta(x, \alpha)),$$

where $\det J_{\alpha}^{\beta}(x, \alpha) = \det \partial_{\alpha} \beta(x, \alpha) = \det_{ij} \partial_{\alpha_i} \beta_j(x, \alpha) \neq 0$.

(ii) Let us check that \mathcal{D} as an element of $C^{\infty}(\Lambda)/C_{\times}^{\infty}(\Lambda)$ is not affected by the reduction of oscillatory variables. We consider the phase function $\phi(x, \lambda, \alpha)$. Assume that the coordinates α split into $\alpha = (\rho, \sigma)$ so that $\phi''_{\sigma\sigma'}$ is non-degenerate. Then there exists a smooth function $\Sigma(x, \rho)$ such that the condition $\phi'_{\sigma}(x, \lambda, \rho, \sigma) = 0$ is equivalent with $\sigma = \Sigma(x, \rho)$. The phase function

$$(A.3) \quad \psi(x, \lambda, \rho) = \phi(x, \lambda, \rho, \Sigma(x, \rho))$$

parametrizes the same canonical relation as ϕ does. We are going to prove that $\det \phi''_{\alpha\alpha'}$ and $\det \psi''_{\rho\rho'}$ differ by a nonzero factor (namely, $\det \phi''_{\sigma\sigma'}$).

In what follows, we drop off the dependence on x and λ . Differentiating (A.3) with respect to ρ , we get

$$(A.4) \quad \psi'_{\rho}(\rho) = \phi'_{\rho}(\rho, \Sigma(\rho)) + \phi'_{\sigma}(\rho, \Sigma(\rho)) J_{\rho}^{\sigma}(\rho),$$

where $J_{\rho}^{\sigma}(\rho) = \partial\Sigma(\rho)/\partial\rho$.

$$(A.5) \quad \begin{aligned} \psi''_{\rho\rho'}(\rho) &= \phi''_{\rho\rho'}(\rho, \Sigma(\rho)) + \phi''_{\rho\sigma}(\rho, \Sigma(\rho)) g_{\rho'}^{\sigma}(\rho) + \phi''_{\rho'\sigma}(\rho, \Sigma(\rho)) J_{\rho}^{\sigma}(\rho) \\ &\quad + \phi''_{\sigma\sigma'}(\rho, \Sigma(\rho)) J_{\rho}^{\sigma}(\rho) J_{\rho'}^{\sigma'}(\rho) + \phi'_{\sigma}(\rho, \Sigma(\rho)) J_{\rho\rho'}^{\sigma}(\rho). \end{aligned}$$

The last term in the right-hand side of (A.5) vanishes identically on the canonical relation (where $\phi'_{\alpha} = (\phi'_{\rho}, \phi'_{\sigma}) \equiv 0$). Using the identity

$$0 \equiv \partial_{\rho}(\phi'_{\sigma}(\rho, \Sigma(\rho))) = \phi''_{\rho\sigma}(\rho, \Sigma(\rho)) + \phi''_{\sigma\sigma'}(\rho, \Sigma(\rho)) J_{\rho}^{\sigma'}(\rho),$$

we can express $J_{\rho}^{\sigma}(\rho) = -\phi''_{\rho\sigma'}(\rho, \Sigma(\rho)) \phi^{\sigma\sigma'}(\rho, \Sigma(\rho))$, where $\phi^{\sigma\sigma'}(\rho, \sigma)$ denotes the matrix inverse to $\phi''_{\sigma\sigma'}(\rho, \sigma)$. We rewrite (A.5) as

$$(A.6) \quad \psi''_{\rho\rho'}(\rho) = \phi''_{\rho\rho'}(\rho, \Sigma(\rho)) - \phi''_{\rho\sigma}(\rho, \Sigma(\rho)) \phi^{\sigma\sigma'}(\rho, \Sigma(\rho)) \phi''_{\sigma'\rho'}(\rho, \Sigma(\rho)).$$

To compute the determinant of (A.6), we use the identity

$$(A.7) \quad \det(A - BD^{-1}C) \det D = \det \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where A and D are square matrices and $\det D \neq 0$, which follows from the matrix identity

$$\begin{bmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{bmatrix} = \begin{bmatrix} I & -BD^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I & 0 \\ -D^{-1}C & I \end{bmatrix}.$$

Identity (A.7) allows us to write the determinant of (A.6) in the form of the desired relation:

$$(A.8) \quad \det \psi''_{\rho\rho'} \det \phi''_{\sigma\sigma'} = \det \phi''_{\alpha\alpha'}, \quad \text{where } \det \phi''_{\sigma\sigma'} \neq 0.$$

This finishes the proof of Lemma 2.7.

APPENDIX B. $\mathfrak{h}^1 \rightarrow L^\infty$ ESTIMATES ON (λ, σ) -PIECES

In this section we prove the following Lemma, needed for sharp $\mathfrak{h}^1 \rightarrow L^q$ and $L^p \rightarrow \text{BMO}$ estimates.

Lemma B.1. *Let $\mathfrak{F} \in I^\mu(X, Y, \mathbf{C})$ be associated to a canonical relation such that $\mathbf{C} \rightarrow X$ is a submersion, and let \mathbf{C} have only caustics of the type A_{m+1} with $m = 1$ or 2 . Then, for any atom a_Q supported in the cube Q with side r , we have*

$$(B.1) \quad \|\Lambda^{-\mu-(n+N)/2} \mathfrak{F}_{\lambda, \sigma} a_Q\|_{L^\infty(X)} \leq \text{const } \sigma^{1/m} \min(\lambda r, (\lambda r)^{-1}).$$

For simplicity, we consider $\mathfrak{F} \in I^\mu(X, Y, \mathbf{C})$ with $\mu = -(N+n)/2$. This implies that $a(x, \theta, y) \in S^d$ with $d = -(n+N)/2 - (N-n)/2 = -N$. Let a_Q be an atom supported in the cube Q with side r (following [SSS91], we may assume that $r \leq 1$). We want to show that for any x ,

$$(B.2) \quad |\mathfrak{F}_{\lambda, \sigma} a_Q(x)| \leq \text{const } \sigma^{1/m} \min(\lambda r, (\lambda r)^{-1}).$$

We will decompose and bound the pieces $\mathfrak{F}_{\lambda, \sigma}$ following the discussion on pages 238-241 in [SSS91]. For a particular λ , we introduce unit vectors θ_λ^ν , with $1 \leq \nu \leq N(\lambda^{-1/2}) \approx \lambda^{\frac{m-1}{2}}$, equidistributed on the unit sphere in the θ -space \mathbb{R}^N , so that $|\theta_\lambda^\nu - \theta_\lambda^{\nu'}| \geq \text{const } \lambda^{-\frac{1}{2}}$ for $\nu \neq \nu'$. We introduce a corresponding partition of unity,

$$1 = \sum_{\nu=1}^{N(\lambda^{-1/2})} \psi_\lambda^\nu(\theta),$$

where the functions ψ_λ^ν are homogeneous of degree 0 and supported in the spherical angles Ω_λ^ν with the span $\sim \lambda^{-1/2}$, centered at θ_λ^ν :

$$\psi_\lambda^\nu(\theta) \neq 0 \quad \text{only if} \quad \left| \frac{\theta}{|\theta|} - \theta_\lambda^\nu \right| \leq \text{const } \lambda^{-\frac{1}{2}}.$$

We assume that $|\partial_\theta^\alpha \psi_\lambda^\nu(\theta)| \leq \text{const } \lambda^{\frac{|\alpha|}{2}} |\theta|^{-|\alpha|}$.

We denote the integral kernels of $\mathfrak{F}_{\lambda, \sigma}$, $\tilde{\mathfrak{F}}_{\lambda, \sigma}$ by $K_{\lambda, \sigma}(x, \theta, y)$ and $\tilde{K}_{\lambda, \sigma}(x, \theta, y)$. We introduce $\mathfrak{F}_{\lambda, \sigma}^\nu$ by

$$\mathfrak{F}_{\lambda, \sigma}^\nu u(x) = \int K_{\lambda, \sigma}^\nu(x, \theta, y) u(y) d\theta dy,$$

where $K_{\lambda, \sigma}^\nu(x, \theta, y) = \psi_\lambda^\nu(\theta) K_{\lambda, \sigma}(x, \theta, y)$.

From now on, we assume that $x \in X$ is fixed. We need to introduce the “exceptional set” associated to x . According to [CC03], the assumption that $\mathbf{C} \rightarrow X$ is a submersion allows one to choose the phase ϕ in the form

$$\phi(x, \theta, y) = \langle G(x, \theta, y'') - y', \theta \rangle,$$

where $y = (y', y'') \in \mathbb{R}^N \times \mathbb{R}^{n-N} = Y$ are certain local coordinates. For given λ and ν , we define $\mathcal{R}_{x, \lambda^{-1}}^\nu \subset Y$ by

$$(B.3) \quad \mathcal{R}_{x, \lambda^{-1}}^\nu = \{y : |\langle G(x, \theta_\lambda^\nu, y'') - y', \theta_\lambda^\nu \rangle| \leq \lambda^{-1}, |G(x, \theta_\lambda^\nu, y'') - y'| \leq \lambda^{-\frac{1}{2}}\},$$

with $|\mathcal{R}_{x, \lambda^{-1}}^\nu| \leq \text{const } \lambda^{-1} \cdot \lambda^{-\frac{m-1}{2}}$. We set $\chi_{\mathcal{R}_{x, \lambda^{-1}}^\nu}(x)$ to be the characteristic function of $\mathcal{R}_{x, \lambda^{-1}}^\nu$.

Let a_Q be an atom supported in the cube Q with side r :

$$|Q| = r^n, \quad \|a_Q\|_{L^\infty} \leq r^{-n}, \quad \|a_Q\|_{L^1} \leq 1, \quad \int_Q a_Q = 0.$$

We consider

$$(B.4) \quad \begin{aligned} \mathfrak{F}_{\lambda, \sigma} a_Q(x) &= \sum_\nu \int \chi_{\mathcal{R}_{x, \lambda^{-1}}^\nu}(y) K_{\lambda, \sigma}^\nu(x, \theta, y) a_Q(y) d\theta dy \\ &+ \sum_\nu \int (1 - \chi_{\mathcal{R}_{x, \lambda^{-1}}^\nu}(y)) K_{\lambda, \sigma}^\nu(x, \theta, y) a_Q(y) d\theta dy. \end{aligned}$$

We need to know the absolute value of this expression.

(i) The absolute value of the first term in the right-hand side is bounded by

$$(B.5) \quad \begin{aligned} &\sum_\nu \int \chi_{\mathcal{R}_{x, \lambda^{-1}}^\nu}(y) |K_{\lambda, \sigma}^\nu(x, \theta, y) a_Q(y)| d\theta dy \\ &\leq C \lambda^{-N} \sum_\nu \int \chi_{\mathcal{R}_{x, \lambda^{-1}}^\nu}(y) \psi_\lambda^\nu(\theta) \beta(|\theta|/\lambda) \beta(\mathcal{D}(x'', \theta, y)/\sigma) |a_Q(y)| d\theta dy. \end{aligned}$$

◦ In (B.5), we have already applied the bound $C \lambda^{-N}$ on the symbol $a(x, \theta, y) \in S^{-N}$ at $|\theta| \sim \lambda$.

◦ Summation in ν converges since $\sum_\nu \psi_\lambda^\nu(\theta) = 1$.

◦ If $d_\theta \mathcal{D} \neq 0$ ($m = 1$), then the integration in θ contributes $\text{const } \sigma \lambda^N$, where σ appears due to the support properties of $\beta(\mathcal{D}/\sigma)$.

More generally, assume that at a point $p \in \mathbf{C}$ there is a simple caustic of the type A_{m+1} with $m = 1$ or 2 . Then there is a vector field $V = a_j \partial_{\theta_j}$, $V \in C^\infty(\Gamma(T(\mathbf{C})))$, such that $V^m \mathcal{D}|_p \neq 0$. We define $\Theta = \frac{\theta}{\lambda} \in \mathbb{R}^N$, so that the region of integration in Θ is bounded uniformly in λ . Note that $d\theta = \lambda^N d\Theta$. We can choose the coordinates so that $\partial_{\theta_N}^m \mathcal{D} \neq 0$, in an open neighborhood of p . The expression $\partial_{\Theta_N}^m \mathcal{D}$ is homogeneous of degree zero in λ , so that $|\partial_{\Theta_N}^m \mathcal{D}| \geq \text{const} > 0$ uniformly in λ, σ . Therefore,

$$\int_{\mathbb{R}} \beta(\mathcal{D}/\sigma) d\Theta_N \leq \text{const } \sigma^{\frac{1}{m}}.$$

The integration in $\Theta_1, \dots, \Theta_{N-1}$ converges since the support of (B.5) in $\Theta = \frac{\theta}{\lambda}$ is bounded (uniformly in λ, σ). We conclude that the integration in θ contributes $\text{const } \lambda^N \sigma^{\frac{1}{m}}$.

◦ Finally, due to the bound $\|a_Q\|_{L^\infty(Y)} \leq |Q|^{-1}$ together with the support properties of a_Q and $\chi_{\mathcal{R}_{x, \lambda^{-1}}^\nu}(y)$, the integral in y contributes the factor $\min(1, (\lambda r)^{-1})$.

Taking the product of all of the above factors, we obtain $\text{const } \sigma^{\frac{1}{m}} \min(1, (\lambda r)^{-1})$.

(ii) For the absolute value of the second term in the right-hand side of (B.4) we have:

◦ In each ν -term, we can integrate by parts as in [SSS91] (we need the assumption $\sigma \geq \lambda^{-\frac{1}{2}}$ to obtain an analogue of the inequalities (3.19) in [SSS91]; the argument is the same as theirs), getting the factor

$$(1 + \lambda^2 |G(x, \theta'_\lambda, y'') - y', \theta'_\lambda|)^2 + \lambda |G(x, \theta'_\lambda, y'') - y'|^2)^{-M}, \quad \text{for any } M \in \mathbb{N}.$$

The integral of the product of this expression with $a_Q(y)$ with respect to y contributes the same factor $\min(1, (\lambda r)^{-1})$ as above. The rest of the analysis is the same as for the first term in the right-hand side of (B.4).

We conclude that $|\mathfrak{F}_{\lambda, \sigma} a_Q(x)| \leq \text{const } \sigma^{\frac{1}{m}} \min(1, (\lambda r)^{-1})$.

The bound in the case $\lambda r < 1$ follows from [CC03]. Let us recall the argument. We fix some point $\bar{y} \in Q$. Since $\int a_Q(y) dy = 0$, we can write

$$\begin{aligned} \mathfrak{F}_{\lambda, \sigma} a_Q(x) &= \int [K_{\lambda, \sigma}(x, \theta, y) - K_{\lambda, \sigma}(x, \theta, \bar{y})] a_Q(y) d\theta dy \\ &= \int_0^1 dt \partial_t \left(\int K_{\lambda, \sigma}(x, \theta, \bar{y} + (y - \bar{y})t) a_Q(y) d\theta dy \right) \\ (B.6) \quad &= \lambda r \int \left\{ \int_0^1 dt \frac{y - \bar{y}}{r} \lambda^{-1} \partial_y K_{\lambda, \sigma}(x, \theta, \bar{y} + (y - \bar{y})t) \right\} a_Q(y) d\theta dy. \end{aligned}$$

The expression in the curly brackets can be treated as an integral kernel of another Fourier integral operator of the same order μ associated to \mathbf{C} , and therefore

$$|\mathfrak{F}_{\lambda, \sigma} a_Q(x)| \leq \lambda r \text{const } \sigma^{1/m}.$$

Let us mention that in (B.6) $|\frac{y - \bar{y}}{r}| \leq \text{const}$ and that the increase in the order of the symbol due to the derivative ∂_y is compensated by λ^{-1} . When the derivative ∂_y acts on $\beta(\mathcal{D}(x, \theta, y)/\sigma)$ (which is hidden inside $K_{\lambda, \sigma}$), the contribution is bounded by $\text{const } \sigma^{-1}$ and is also compensated by λ^{-1} . The integration in t is irrelevant.

This completes the proof of Lemma B.1.

ACKNOWLEDGMENT

The author would like to thank Michael Taylor for calling the author's attention to the subject of caustics and for his continuous interest, help, and encouragement.

REFERENCES

- [AGZV88] V. I. Arnol'd, S. M. Gusein-Zade, and A. N. Varchenko, *Singularities of differentiable maps. Vol. II*, Birkhäuser Boston Inc., Boston, MA, 1988. MR **89g**:58024
- [Bre75] Philip Brenner, *On $L_p - L_{p'}$ estimates for the wave-equation*, Math. Z. **145** (1975), no. 3, 251–254. MR **52**:8658
- [Bre77] ———, *$L_p - L_{p'}$ -estimates for Fourier integral operators related to hyperbolic equations*, Math. Z. **152** (1977), no. 3, 273–286. MR **55**:3877
- [CC03] Andrew Comech and Scipio Cuccagna, *On L^p continuity of singular Fourier integral operators*, Trans. Amer. Math. Soc. **355** (2003), no. 6, 2453–2476.
- [CCW99] Anthony Carbery, Michael Christ, and James Wright, *Multidimensional van der Corput and sublevel set estimates*, J. Amer. Math. Soc. **12** (1999), no. 4, 981–1015. MR **2000h**:42010
- [CdV77] Y. Colin de Verdière, *Nombre de points entiers dans une famille homothétique de domaines de \mathbf{r}* , Ann. Sci. École Norm. Sup. (4) **10** (1977), no. 4, 559–575. MR **58**:563
- [Dui74] J. J. Duistermaat, *Oscillatory integrals, Lagrange immersions and unfolding of singularities*, Comm. Pure Appl. Math. **27** (1974), 207–281. MR **53**:9306

- [Dui96] ———, *Fourier integral operators*, Birkhäuser Boston Inc., Boston, MA, 1996. MR **96m**:58245
- [GS77] Victor Guillemin and Shlomo Sternberg, *Geometric asymptotics*, American Mathematical Society, Providence, R.I., 1977, Mathematical Surveys, No. 14. MR **58**:24404
- [GS02] Allan Greenleaf and Andreas Seeger, *Oscillatory and Fourier integral operators with degenerate canonical relations*, Proceedings of the 6th International Conference on Harmonic Analysis and Partial Differential Equations (El Escorial, 2000), no. Vol. Extra, 2002, pp. 93–141.
- [GSW00] Allan Greenleaf, Andreas Seeger, and Stephen Wainger, *Estimates for generalized Radon transforms in three and four dimensions*, Analysis, geometry, number theory: the mathematics of Leon Ehrenpreis (Philadelphia, PA, 1998), Contemp. Math., vol. 251, Amer. Math. Soc., Providence, RI, 2000, pp. 243–254. MR **2001j**:58047
- [Hör71] Lars Hörmander, *Fourier integral operators. I*, Acta Math. **127** (1971), no. 1-2, 79–183. MR **52**:9299
- [Hör94] ———, *The analysis of linear partial differential operators. III*, Springer-Verlag, Berlin, 1994, Pseudo-differential operators, Corrected reprint of the 1985 original. MR **95h**:35255
- [JMR00] Jean-Luc Joly, Guy Metivier, and Jeffrey Rauch, *Caustics for dissipative semilinear oscillations*, Mem. Amer. Math. Soc. **144** (2000), no. 685, viii+72. MR **2000i**:35115
- [Lit73] Walter Littman, *$L^p - L^q$ -estimates for singular integral operators arising from hyperbolic equations*, Partial differential equations (Proc. Sympos. Pure Math., Vol. XXIII, Univ. California, Berkeley, Calif., 1971), Amer. Math. Soc., Providence, R.I., 1973, pp. 479–481. MR **50**:10909
- [Lud66] Donald Ludwig, *Uniform asymptotic expansions at a caustic*, Comm. Pure Appl. Math. **19** (1966), 215–250. MR **33**:4446
- [Mag01] Akos Magyar, *Estimates for the wave kernel near focal points on compact manifolds*, J. Geom. Anal. **11** (2001), no. 1, 119–128. MR **2002d**:58035
- [MT85] Richard B. Melrose and Michael E. Taylor, *Near peak scattering and the corrected Kirchhoff approximation for a convex obstacle*, Adv. in Math. **55** (1985), no. 3, 242–315. MR **86m**:35095
- [PS91] D. H. Phong and E. M. Stein, *Radon transforms and torsion*, Internat. Math. Res. Notices (1991), no. 4, 49–60. MR **93g**:58144
- [See93] Andreas Seeger, *Degenerate Fourier integral operators in the plane*, Duke Math. J. **71** (1993), no. 3, 685–745. MR **94h**:35292
- [Sog93] Christopher D. Sogge, *Fourier integrals in classical analysis*, Cambridge University Press, Cambridge, 1993. MR **94c**:35178
- [SS94] Hart F. Smith and Christopher D. Sogge, *L^p regularity for the wave equation with strictly convex obstacles*, Duke Math. J. **73** (1994), no. 1, 97–153. MR **95c**:35048
- [SSS91] Andreas Seeger, Christopher D. Sogge, and Elias M. Stein, *Regularity properties of Fourier integral operators*, Ann. of Math. (2) **134** (1991), no. 2, 231–251. MR **92g**:35252
- [Str70] Robert S. Strichartz, *Convolutions with kernels having singularities on a sphere*, Trans. Amer. Math. Soc. **148** (1970), 461–471. MR **41**:876
- [Sug94] Mitsuru Sugimoto, *A priori estimates for higher order hyperbolic equations*, Math. Z. **215** (1994), no. 4, 519–531. MR **95j**:35128
- [Sug96] ———, *Estimates for hyperbolic equations with non-convex characteristics*, Math. Z. **222** (1996), no. 4, 521–531. MR **97f**:35122
- [Sug98] ———, *Estimates for hyperbolic equations of space dimension 3*, J. Funct. Anal. **160** (1998), no. 2, 382–407. MR **2000i**:35113
- [Tom79] Peter A. Tomas, *Restriction theorems for the Fourier transform*, Harmonic analysis in Euclidean spaces (Proc. Sympos. Pure Math., Williams Coll., Williamstown, Mass., 1978), Part 1, Amer. Math. Soc., Providence, R.I., 1979, pp. 111–114. MR **81d**:42029

DEPARTMENT OF MATHEMATICS, DUKE UNIVERSITY, DURHAM, NORTH CAROLINA 27708
E-mail address: comech@math.duke.edu